114
51

TRANSACTIONS

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

ELIAKIM HASTINGS MOORE

ERNEST WILLIAM BROWN THOMAS SCOTT FISKE

PUBLISHED QUARTERLY BY THE SOCIETY WITH
THE COÖPERATION OF

HARVARD UNIVERSITY
YALE UNIVERSITY
PRINCETON UNIVERSITY
COLUMBIA UNIVERSITY
HAVERFORD COLLEGE
NORTHWESTERN UNIVERSITY
CORNELL UNIVERSITY
THE UNIVERSITY OF CALIFORNIA
BRYN MAWR COLLEGE
THE UNIVERSITY OF CHICAGO

VOLUME 2
1901

LANCASTER, PA., AND NEW YORK
THE MACMILLAN COMPANY
AGENTS FOR THE SOCIETY
1901

PRESS OF
THE NEW ERA PRINTING COMPANY,
LANCASTER, PA.

33 2040

TABLE OF CONTENTS

VOLUME 2 1901

PAGES

BLICHFELDT, H. F., of Stanford University, Cal. Note on the functions of the form

$$f(x) \equiv \phi(x) + d_1 x^{n-1} + d_2 x^{n-2} + \cdots + d_n$$

which in a given interval differ the least possible from zero . . 100–102

———— A new determination of the primitive continuous groups in two variables 249–258

BÔCHER, MAXIME, of Cambridge, Mass. Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence 139–149

———— An elementary proof of a theorem of Sturm 150–151

———— On certain pairs of transcendental functions whose roots separate each other 428–436

BOLZA, O., of Chicago, Ill. New proof of a theorem of Osgood's in the calculus of variations 422–427

CAJORI, FLORIAN, of Colorado Springs, Col. Divergent and conditionally convergent series whose product is absolutely convergent 25– 36

DICKSON, LEONARD EUGENE, of Chicago, Ill. Canonical forms of quaternary abelian substitutions in an arbitrary Galois field . 103–138

———— Theory of linear groups in an arbitrary field 363–394

FIELDS, J. C., of Hamilton, Ontario, Canada. On the reduction of the general abelian integral 49– 86

HILBERT, DAVID, of Göttingen, Germany. Ueber Flächen von constanter Gaussscher Krümmung 87– 99

MCDONALD, J. H., of Lafayette, Ind. On the system of a binary cubic and quadratic and the reduction of hyperelliptic integrals of genus two to elliptic integrals by a transformation of the fourth order 437–458

METZLER, W. H., of Syracuse, N. Y. On certain aggregates of determinant minors 395–403

MERTENS, F., of Vienna, Austria. Zur linearen Transformation der ϑ-Reihen 331–342

MILLER, G. A., of Ithaca, N. Y. Determination of all the groups of order p^m which contain the abelian groups of type $(m-2, 1)$, p being any prime 259–272

Math. 11-1902 Macmillan 5.00

50037

PAGES

MOORE, ELIAKIM HASTINGS, of Chicago, Ill. Concerning Harnack's theory of improper definite integrals 296–330

———— On the theory of improper definite integrals 459–475

OSGOOD, W. F., of Cambridge, Mass. On the existence of a minimum of the integral

$$\int_{x_0}^{x_1} F(x, y, y')\,dx$$

when x_0 and x_1 are conjugate points, and the geodesics on an ellipsoid of revolution: a revision of a theorem of Kneser's . . 166–182

———— On a fundamental property of a minimum in the calculus of variations and the proof of a theorem of Weierstrass's 273–295

PORTER, M. B., of New Haven, Conn. Sets of coincidence points on the non-singular cubics of a syzygetic sheaf 37– 42

PRINGSHEIM, A., of Munich, Germany. Ueber die Anwendung der Cauchy'schen Multiplicationsregel auf bedingt convergente oder divergente Reihen 404–412

———— Ueber den Goursat'schen Beweis des Cauchy'schen Integralsatzes 413–421

SMITH, PERCEY F., of New Haven, Conn. Geometry within a linear spherical complex 234–248

STECKER, HENRY FREEMAN, of Ithaca, N. Y. On the determination of surfaces capable of conformal representation upon the plane in such a manner that geodetic lines are represented by algebraic curves 152–165

STRINGHAM, IRVING, of Berkeley, Cal. On the geometry of planes in a parabolic space of four dimensions 183–214

STRONG, WENDELL M., of New Haven, Conn. Note on non-quaternion number systems 43– 48

VAN VLECK, E. B., of Middletown, Conn. On the convergence of continued fractions with complex elements 215–233

———— On the convergence and character of a certain form of continued fraction 476–483

WILCZYNSKI, E. J., of Berkeley, Cal. Invariants of systems of linear differential equations 1– 24

———— Geometry of a simultaneous system of two linear homogeneous differential equations of the second order 343–362

Notes and errata: volumes 1 and 2 484–487

LIBRARY
OF THE
UNIVERSITY of ILLINOIS.

INVARIANTS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

BY

E. J. WILCZYNSKI

The theory of the invariants of a single linear homogeneous differential equation rests upon STAECKEL'S theorem, that the most general point-transformation which converts a general homogeneous linear differential equation of the m-th order $(m > 1)$

$$\frac{d^m y}{dx^m} + P_1(x)\frac{d^{m-1}y}{dx^{m-1}} + \cdots + P_m(x)y = 0, \tag{1}$$

into another equation of the same form and order, is

$$x = f(\xi), \quad y = \eta\, g(\xi), \tag{2}$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions of ξ.*

Those functions of $P_1, P_2, \cdots, P_m$ and the derivatives of these quantities, which are the same for the equation (1) and for any equation obtained from (1) by a transformation (2), are called invariants of (1). Functions having this invariant property and containing also y, dy/dx, etc., are called covariants. The investigation of such invariants and covariants has led to many new and interesting results concerning the equation (1). This theory is associated with the names of COCKLE, MALET, LAGUERRE, HALPHEN, BRIOSCHI, FORSYTH, and others.

The author has recently shown that the most general point-transformation, which converts a system of n homogeneous linear differential equations into another of the same form and order, is

$$x = f(\xi), \quad y_k = \sum_{i=1}^{n} a_{ki}(\xi)\eta_i \qquad (k = 1, 2, \cdots, n), \tag{3}$$

where $f(\xi)$ and $a_{ki}(\xi)$ are arbitrary functions of ξ, and the determinant $|a_{ki}(\xi)|$ does not vanish identically.†

We shall consider, in this paper, those combinations of the coefficients of a system of linear differential equations which remain invariant when the system

*Crelle's Journal, vol. 111.

†American Journal of Mathematics, January, 1901.

50037

is transformed by the above transformation (3). These transformations obviously form an infinite continuous group, and we shall employ LIE's theory throughout, as has been done in the case of a single differential equation by Dr. BOUTON.* We shall not, in this first paper, pay much attention to the applications of the theory, nor give more than a passing mention to covariants.

§ 1. *Finite transformations of the dependent variables.*

We shall at first confine ourselves to the transformation of the dependent variables. Those functions of the coefficients of the system which remain invariant for all such transformations, may be called *seminvariants*. These are of considerable importance in themselves, and besides furnish the basis for the theory of invariants under the general transformation.

Let the given system be

$$y_i^{(m)} + \sum_{l=0}^{m-1} \sum_{k=1}^{n} p_{ikl} y_k^{(l)} = 0 \qquad (i = 1, 2, \cdots, n), \tag{1}$$

and let this system be transformed by the equations

$$y_k = \sum_{\lambda=1}^{n} a_{k\lambda}(x)\eta_\lambda \qquad (k = 1, 2, \cdots, n), \tag{2}$$

where $a_{k\lambda}(x)$ are arbitrary functions of x, and where the determinant

$$|a_{k\lambda}(x)| \qquad (k, \lambda = 1, 2, \cdots, n),$$

does not vanish. As usual we denote derivatives by accents. Then we have from (2)

$$y_k^{(l)} = \sum_{\lambda=1}^{n} \sum_{\rho=0}^{l} \binom{l}{\rho} a_{k\lambda}^{(\rho)} \eta_\lambda^{(l-\rho)} \qquad (k = 1, 2, \cdots, n;\ l = 0, 1, 2, \cdots, m), \tag{3}$$

where in general $\binom{l}{\rho}$ denotes the coefficient of x^ρ in the expansion of $(1 + x)^l$.

Equations (1) then become

$$\sum_{\lambda=1}^{n} a_{i\lambda}\eta_\lambda^{(m)} + \sum_{\lambda=1}^{n} \sum_{\rho=1}^{m} \binom{m}{\rho} a_{i\lambda}^{(\rho)} \eta_\lambda^{(m-\rho)} + \sum_{l=0}^{m-1} \sum_{k=1}^{n} \sum_{\mu=1}^{n} \sum_{\sigma=0}^{l} \binom{l}{\sigma} p_{ikl} a_{k\mu}^{(\sigma)} \eta_\mu^{(l-\sigma)} = 0 \tag{4}$$

$$(i = 1, 2, \cdots, n).$$

The coefficient of $\eta_\mu^{(\nu)}$ in the double sum is

$$\binom{m}{m-\nu} a_{i\mu}^{(m-\nu)},$$

and in the quadruple sum, the coefficient of $\eta_\mu^{(\nu)}$ is

*American Journal of Mathematics, vol. 21, no. 2, 1899.

$$\sum_{k=1}^{n}\left[\binom{\nu}{0}p_{ik\nu}a_{k\mu}+\binom{\nu+1}{1}p_{i,k,\nu+1}a_{k\mu}^{(1)}+\cdots+\binom{m-1}{m-1-\nu}p_{i,k,m-1}a_{k\mu}^{(m-1-\nu)}\right],$$

i. e.,

$$\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{i,k,\nu+\tau}a_{k\mu}^{(\tau)}.$$

Thus, from (4) we obtain

$$(5)\quad \sum_{\lambda=1}^{n}a_{i\lambda}\eta_{\lambda}^{(m)}+\sum_{\mu=1}^{n}\sum_{\nu=0}^{m-1}\eta_{\mu}^{(\nu)}\left[\binom{m}{m-\nu}a_{i\mu}^{(m-\nu)}+\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{i,k,\nu+\tau}a_{k\mu}^{(\tau)}\right]=0 \qquad (i=1,2,\cdots,n),$$

or if we put

$$(6)\quad \Delta=|a_{i\lambda}| \qquad (i,\lambda=1,2,\cdots,n),$$

and denote by $A_{i\lambda}$ the minor of $a_{i\lambda}$ in this determinant,

$$(7)\quad \Delta\eta_{\lambda}^{(m)}+\sum_{\mu=1}^{n}\sum_{\nu=0}^{m-1}\eta_{\mu}^{(\nu)}\sum_{i=1}^{n}A_{i\lambda}\left[\binom{m}{m-\nu}a_{i\mu}^{(m-\nu)}+\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{i,k,\nu+\tau}a_{k\mu}^{(\tau)}\right]=0 \qquad (\lambda=1,2,\cdots,n).$$

If then we write this system in the form

$$(8)\quad \eta_{\lambda}^{(m)}+\sum_{\nu=0}^{m-1}\sum_{\mu=1}^{n}\pi_{\lambda\mu\nu}\eta_{\mu}^{(\nu)}=0 \qquad (\lambda=1,2,\cdots,n),$$

we have

$$(9)\quad \Delta\pi_{\lambda\mu\nu}=\sum_{i=1}^{n}A_{i\lambda}\left[\binom{m}{m-\nu}a_{i\mu}^{(m-\nu)}+\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{i,k,\nu+\tau}a_{k\mu}^{(\tau)}\right]$$
$$(\lambda,\mu=1,2,\cdots,n;\ \nu=0,1,\cdots,m-1).$$

Thus, if (1) is transformed into (8) by transformations (2), the relations between the coefficients of (1) and (8) are the equations (9).

Equations (9) represent an infinite continuous group, isomorphic with the group represented by equations (2). For to every transformation of the latter group corresponds one of the former, and they obviously have the group property. Both groups can be defined by differential equations, so that LIE'S theory of infinite groups may be applied.

§ 2. *Infinitesimal transformations of the dependent variables.*

We proceed to consider the infinitesimal transformations of our infinite group. The variables $y_1, y_2, \cdots, y_n$ will undergo the most general infinitesimal transformation of form (2) if we put

$$a_{ii}(x) = 1 + \phi_{ii}(x)\delta t, \quad a_{ik}(x) = \phi_{ik}(x)\delta t \tag{10}$$
$$(i \neq k),\ (i, k = 1, 2, \cdots, n),$$

where δt is an infinitesimal and the ϕ_{ik}'s are arbitrary functions of x. We wish to find the corresponding infinitesimal transformations of the coefficients p_{ikl}.

Neglecting infinitesimals of order higher than the first, we find

$$\Delta = \begin{vmatrix} 1 + \phi_{11}\delta t, & \phi_{12}\delta t, & \cdots, & \phi_{1n}\delta t \\ \phi_{21}\delta t, & 1 + \phi_{22}\delta t, & \cdots, & \phi_{2n}\delta t \\ \cdot & \cdot & \cdot & \cdot \\ \phi_{n1}\delta t, & \phi_{n2}\delta t, & \cdots, & 1 + \phi_{nn}\delta t \end{vmatrix} = 1 + (\phi_{11} + \phi_{22} + \cdots + \phi_{nn})\delta t, \tag{11}$$

and similarly

$$\begin{cases} A_{ii} = 1 + (\phi_{11} + \phi_{22} + \cdots + \phi_{nn} - \phi_{ii})\delta t, \\ A_{ki} = -\phi_{ik}\delta t \end{cases} \qquad (i \neq k). \tag{12}$$

Substituting these values in (9), we have

$$\Delta\pi_{\lambda\mu\nu} = \sum_{i=1}^{n} -\phi_{\lambda i}\delta t\left[\binom{m}{m-\nu}\phi_{i\mu}^{(m-\nu)}\delta t + \sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{i,k,\nu+\tau}\phi_{k\mu}^{(\tau)}\delta t + p_{i\mu\nu}\right] + [1 + (\phi_{11} + \phi_{22} + \cdots + \phi_{nn})\delta t]\left[\binom{m}{m-\nu}\phi_{\lambda\mu}^{(m-\nu)}\delta t + \sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{\lambda,k,\nu+\tau}\phi_{k\mu}^{(\tau)}\delta t + p_{\lambda\mu\nu}\right],$$

or

$$\Delta\pi_{\lambda\mu\nu} = p_{\lambda\mu\nu} - \sum_{i=1}^{n}\phi_{\lambda i}p_{i\mu\nu}\delta t + \binom{m}{m-\nu}\phi_{\lambda\mu}^{(m-\nu)}\delta t + \sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{\lambda,k,\nu+\tau}\phi_{k\mu}^{(\tau)}\delta t + p_{\lambda\mu\nu}(\phi_{11} + \phi_{22} + \cdots + \phi_{nn})\delta t. \tag{13}$$

Dividing by $\Delta = 1 + (\phi_{11} + \cdots + \phi_{nn})\delta t$, and denoting the infinitesimal difference $\pi_{\lambda\mu\nu} - p_{\lambda\mu\nu}$ by $\delta p_{\lambda\mu\nu}$, we find

$$\frac{\delta p_{\lambda\mu\nu}}{\delta t} = \sum_{k=1}^{n}(\phi_{k\mu}p_{\lambda k\nu} - \phi_{\lambda k}p_{k\mu\nu}) + \sum_{k=1}^{n}\sum_{\tau=1}^{m-1-\nu}\binom{\nu+\tau}{\tau}\phi_{k\mu}^{(\tau)}p_{\lambda,k,\nu+\tau} + \binom{m}{m-\nu}\phi_{\lambda\mu}^{(m-\nu)}, \tag{14}$$
$$(\lambda, \mu = 1, 2, \cdots, n;\ \nu = 0, 1, 2, \cdots, m-1).$$

These are the required infinitesimal transformations of $p_{\lambda\mu\nu}$. Those of $p'_{\lambda\mu\nu}$, $p''_{\lambda\mu\nu}$, etc., may be obtained from (14) by differentiation.

§ 3. *Calculation of the seminvariants for* $m = n = 2$.

The complexity of the general problem is so great that it appears wise to limit the further discussion in this paper to the very special case $m = n = 2$. This will throw considerable light on the general case.

Let us put, in this case, for abbreviation

$$(15)\qquad \begin{cases} p_{\lambda,\mu,m-1} = p_{\lambda\mu 1} = p_{\lambda\mu}, \qquad p_{\lambda,\mu,m-2} = p_{\lambda\mu 0} = q_{\lambda\mu}, \\ \dfrac{\partial f}{\partial p_{\lambda\mu}} = P_{\lambda\mu}, \quad \dfrac{\partial f}{\partial p'_{\lambda\mu}} = P'_{\lambda\mu}, \quad \dfrac{\partial f}{\partial q_{\lambda\mu}} = Q_{\lambda\mu}, \quad \text{etc.} \end{cases}$$

We have then, from (14),

$$(16)\qquad \begin{cases} \dfrac{\delta p_{\lambda\mu}}{\delta t} = \sum_{k=1}^{2}(\phi_{k\mu}p_{\lambda k} - \phi_{\lambda k}p_{k\mu}) + 2\phi'_{\lambda\mu}, \\ \dfrac{\delta p'_{\lambda\mu}}{\delta t} = \sum_{k=1}^{2}(\phi_{k\mu}p'_{\lambda k} - \phi_{\lambda k}p'_{k\mu} + \phi'_{k\mu}p_{\lambda k} - \phi'_{\lambda k}p_{k\mu}) + 2\phi^{(2)}_{\lambda\mu}, \\ \dfrac{\delta q_{\lambda\mu}}{\delta t} = \sum_{k=1}^{2}(\phi_{k\mu}q_{\lambda k} - \phi_{\lambda k}q_{k\mu}) + \sum_{k=1}^{2}\phi'_{k\mu}p_{\lambda k} + \phi^{(2)}_{\lambda\mu}. \end{cases}$$

Now if f is a seminvariant depending only upon $p_{\lambda\mu}$, $p'_{\lambda\mu}$, $q_{\lambda\mu}$, we must have

$$(17)\qquad \sum_{\lambda,\mu}(P_{\lambda\mu}\delta p_{\lambda\mu} + P'_{\lambda\mu}\delta p'_{\lambda\mu} + Q_{\lambda\mu}\delta q_{\lambda\mu}) = 0$$

for all values of ϕ_{rs}, ϕ'_{rs}, ϕ''_{rs}. Putting the coefficients of these twelve arbitrary functions equal to zero, we obtain the following system of partial differential equations for such seminvariants:

$$(18)\qquad \begin{cases} 2P'_{rs} + Q_{rs} = 0, \\ 2P_{rs} + \sum_{\lambda=1}^{2}(p_{\lambda r}P'_{\lambda s} - p_{s\lambda}P'_{r\lambda} + p_{\lambda r}Q_{\lambda s}) = 0, \qquad (r, s = 1, 2). \\ \sum_{\lambda=1}^{2}(p_{\lambda r}P_{\lambda s} - p_{s\lambda}P_{r\lambda} + p'_{\lambda r}P'_{\lambda s} - p'_{s\lambda}P'_{r\lambda} + q_{\lambda r}Q_{\lambda s} - q_{s\lambda}Q_{r\lambda}) = 0. \end{cases}$$

This is a complete system of twelve equations with twelve independent variables. But there are two relations among them, so that we shall have two independent solutions, i. e., two independent seminvariants containing only the variables $p_{\lambda\mu}$, $p'_{\lambda\mu}$, $q_{\lambda\mu}$.

The first four equations of the system tell us that the quantities p'_{rs} and q_{rs} can occur only in combinations

$$2p'_{rs} - 4q_{rs}.$$

The next four equations, written separately, are

$$(19)\quad \begin{cases} 2P_{12}+(p_{11}-p_{22})P'_{12}+p_{21}(P'_{22}-P'_{11})+p_{11}Q_{12}+p_{21}Q_{22}=0\,, \\ 2P_{11}+\ p_{21}P'_{21}-p_{12}P'_{12}+p_{11}Q_{11}+p_{21}Q_{21}=0\,, \\ 2P_{22}+\ p_{12}P'_{12}-p_{21}P'_{21}+p_{12}Q_{12}+p_{22}Q_{22}=0\,, \\ 2P_{21}+(p_{22}-p_{11})P'_{21}+p_{12}(P'_{11}-P'_{22})+p_{12}Q_{11}+p_{22}Q_{21}=0\,. \end{cases}$$

They show that the only possible combinations of p_{rs}, p'_{rs}, q_{rs} are

$$(20)\quad \begin{cases} u_{11}=2p'_{11}-4q_{11}+p_{11}^2+p_{12}p_{21}\,, \\ u_{12}=2p'_{12}-4q_{12}+p_{12}(p_{11}+p_{22})\,, \\ u_{21}=2p'_{21}-4q_{21}+p_{21}(p_{11}+p_{22})\,, \\ u_{22}=2p'_{22}-4q_{22}+p_{22}^2+p_{12}p_{21}\,, \end{cases}$$

so that the seminvariants, here considered, are functions of u_{11}, u_{12}, u_{21}, u_{22} only.

The last four equations (18) are

$$(21)\quad \begin{cases} U_1\equiv (p_{11}-p_{22})P_{12}+p_{21}(P_{22}-P_{11})+(p'_{11}-p'_{22})P'_{12} \\ \qquad\qquad +p'_{21}(P'_{22}-P'_{11})+(q_{11}-q_{22})Q_{12}+q_{21}(Q_{22}-Q_{11})=0\,, \\ U_2\equiv\ p_{21}P_{21}-p_{12}P_{12}+p'_{21}P'_{21}-p'_{12}P'_{12}+q_{21}Q_{21}-q_{12}Q_{12}=0\,, \\ U_3\equiv\ p_{12}P_{12}-p_{21}P_{21}+p'_{12}P'_{12}-p'_{21}P'_{21}+q_{12}Q_{12}-q_{21}Q_{21}=0\,, \\ U_4\equiv (p_{22}-p_{11})P_{21}+p_{12}(P_{11}-P_{22})+(p'_{22}-p'_{11})P'_{21} \\ \qquad\qquad +p'_{12}(P'_{11}-P'_{22})+(q_{22}-q_{11})Q_{21}+q_{12}(Q_{11}-Q_{22})=0\,, \end{cases}$$

with the obvious relation

$$(22)\qquad U_2+U_3=0\,.$$

But there is another relation between these four equations and the other eight, these latter being themselves obviously independent. If we compute from them P_{rs} and P'_{rs} and substitute the values of these quantities, thus obtained, in (21), that system becomes

$$(23)\quad \begin{cases} (u_{11}-u_{22})Q_{12}-u_{21}(Q_{11}-Q_{22})=0\,, \\ -(u_{11}-u_{22})Q_{21}+u_{12}(Q_{11}-Q_{22})=0\,, \\ u_{12}Q_{12}-u_{21}Q_{21}=0\,, \end{cases}$$

where the last equation is a consequence of the other two.

The two independent solutions of this system are

$$I = u_{11} + u_{22}, \quad J = u_{11}u_{22} - u_{12}u_{21}, \tag{24}$$

and these are the seminvariants required.

Let us proceed to obtain next those seminvariants which contain also the quantities p''_{ik} and q'_{ik}. They must satisfy the following system of partial differential equations:

$$\begin{cases} (a)\ 2P''_{rs} + Q'_{rs} = 0, \\ (b)\ 2P'_{rs} + Q_{rs} + \sum_{\lambda=1}^{2} (p_{\lambda r}P''_{\lambda s} - p_{s\lambda}P''_{r\lambda} + p_{\lambda r}Q'_{\lambda s}) = 0, \\ (c)\ 2P_{rs} + \sum_{\lambda=1}^{2} (p_{\lambda r}P'_{\lambda s} - p_{s\lambda}P'_{r\lambda} - 2p'_{s\lambda}P''_{r\lambda} \\ \qquad + q_{\lambda r}Q'_{\lambda s} - q_{s\lambda}Q'_{r\lambda} + p_{\lambda r}Q_{\lambda s}) = 0, \\ (d)\ \sum_{\lambda=1}^{2} (p_{\lambda r}P_{\lambda s} - p_{s\lambda}P_{r\lambda} + p'_{\lambda r}P'_{\lambda s} - p'_{s\lambda}P'_{r\lambda} + p''_{\lambda r}P''_{\lambda s} \\ \qquad - p''_{s\lambda}P''_{r\lambda} + q_{\lambda r}Q_{\lambda s} - q_{s\lambda}Q_{r\lambda} + q'_{\lambda r}Q'_{\lambda s} - q'_{s\lambda}Q'_{r\lambda}) = 0, \end{cases} \tag{25}$$

among which there is one relation. Thus there are 15 independent equations and 20 variables. There are therefore five seminvariants satisfying these equations. Of these we already know four, namely, I, J, dI/dx, dJ/dx, which are obviously independent.

Let us put

$$u'_{ik} = \frac{du_{ik}}{dx}, \quad \text{etc.}$$

Then, since according to (20), we have

$$u_{ik} = 2p'_{ik} - 4q_{ik} + \sum_{j=1}^{2} p_{ij}p_{jk}, \tag{26}$$

therefore

$$u'_{ik} = 2p''_{ik} - 4q'_{ik} + \sum_{j=1}^{2} (p_{ij}p'_{jk} + p'_{ij}p_{jk}). \tag{27}$$

It will be easily proved from equations (25) (a) and (b) that our seminvariants are functions of the twelve arguments

$$p_{ik}, \quad u_{ik}, \quad u'_{ik}. \tag{28}$$

Denoting the left members of (25) (c) by $\Omega_1 \cdots \Omega_4$, so that

$$\Omega_1 = 2P_{11} + \cdots, \quad \Omega_2 = 2P_{12} + \cdots, \quad \Omega_3 = 2P_{21} + \cdots, \quad \Omega_4 = 2P_{22} + \cdots,$$

we find

$$\Omega_1(u_{ik}) = \Omega_2(u_{ik}) = \Omega_3(u_{ik}) = \Omega_4(u_{ik}) = 0, \tag{29}$$

$$\begin{cases} \Omega_1(p_{11}) = 2, & \Omega_2(p_{11}) = 0, & \Omega_3(p_{11}) = 0, & \Omega_4(p_{11}) = 0, \\ \Omega_1(p_{12}) = 0, & \Omega_2(p_{12}) = 2, & \Omega_3(p_{12}) = 0, & \Omega_4(p_{12}) = 0, \\ \Omega_1(p_{21}) = 0, & \Omega_2(p_{21}) = 0, & \Omega_3(p_{21}) = 2, & \Omega_4(p_{21}) = 0, \\ \Omega_1(p_{22}) = 0, & \Omega_2(p_{22}) = 0, & \Omega_3(p_{22}) = 0, & \Omega_4(p_{22}) = 2, \end{cases} \tag{30}$$

and

$$(31)\quad \begin{cases} \Omega_1(u'_{11}) = 0\,, & \Omega_2(u'_{11}) = -u_{21}\,, & \\ & \Omega_3(u'_{11}) = +u_{12}\,, & \Omega_4(u'_{11}) = 0\,, \\ \Omega_1(u'_{12}) = -u_{12}\,, & \Omega_2(u'_{12}) = u_{11} - u_{22}\,, & \\ & \Omega_3(u'_{12}) = 0\,, & \Omega_4(u'_{12}) = +u_{12}\,, \\ \Omega_1(u'_{21}) = +u_{21}\,, & \Omega_2(u'_{21}) = 0\,, & \\ & \Omega_3(u'_{21}) = -(u_{11} - u_{22})\,, & \Omega_4(u'_{21}) = -u_{21}\,, \\ \Omega_1(u'_{22}) = 0\,, & \Omega_2(u'_{22}) = +u_{21}\,, & \\ & \Omega_3(u'_{22}) = -u_{12}\,, & \Omega_4(u'_{22}) = 0\,. \end{cases}$$

From these equations it is easily seen that the eight independent functions of the arguments (28) which verify the equations $\Omega_\lambda = 0$, are the quantities u_{ik} and

$$(32)\quad \begin{cases} v_{11} = 2u'_{11} + p_{12}u_{21} - p_{21}u_{12}\,, \\ v_{12} = 2u'_{12} + (p_{11} - p_{22})u_{12} - p_{12}(u_{11} - u_{22})\,, \\ v_{21} = 2u'_{21} - (p_{11} - p_{22})u_{21} + p_{21}(u_{11} - u_{22})\,, \\ v_{22} = 2u'_{22} - p_{12}u_{21} + p_{21}u_{12}\,. \end{cases}$$

Denoting by $X_1, \cdots, X_4$ the left members of (25) (d) we find:

$$(33)\quad \begin{cases} X_1(v_{11}) = 0\,, & X_2(v_{11}) = -v_{21}\,, & \\ & X_3(v_{11}) = +v_{12}\,, & X_4(v_{11}) = 0\,, \\ X_1(v_{12}) = -v_{12}\,, & X_2(v_{12}) = v_{11} - v_{22}\,, & \\ & X_3(v_{12}) = 0\,, & X_4(v_{12}) = +v_{12}\,, \\ X_1(v_{21}) = +v_{21}\,, & X_2(v_{21}) = 0\,, & \\ & X_3(v_{21}) = -(v_{11} - v_{22})\,, & X_4(v_{21}) = -v_{21}\,, \\ X_1(v_{22}) = 0\,, & X_2(v_{22}) = +v_{21}\,, & \\ & X_3(v_{22}) = -v_{12}\,, & X_4(v_{22}) = 0\,, \end{cases}$$

the equations for $X_1(u_{ik})$, etc., being precisely of the same form.

From (33) we find that

$$v_{11} + v_{22} \quad \text{and} \quad v_{11}v_{22} - v_{12}v_{21}$$

are solutions of the equations $X_\lambda = 0$. But

$$(34)\qquad v_{11} + v_{22} = 2I'\,,$$

while

$$K = v_{11}v_{22} - v_{12}v_{21}, \tag{35}$$

is obviously a new seminvariant, independent of I, J, dI/dx, dJ/dx.

If we should now write down the differential equations satisfied by the seminvariants involving $p_{ik}^{(3)}$ and $q_{ik}^{(2)}$ besides the quantities already considered, there would be twenty such equations with only one relation between them, and twenty-eight independent variables. Hence there must be $28 - 19 = 9$ such seminvariants. But we know eight of these, viz.:

$$I,\ I',\ I'';\quad J,\ J',\ J'';\quad K,\ K'; \tag{36}$$

these are independent, for it is easily seen that from the existence of a relation between them would follow the existence of a relation between I, I', J, J', K. But these quantities were independent.

We can obtain the ninth semivariant without writing down and integrating the last mentioned system of twenty equations. The process which we shall employ is much more instructive, and is capable of generalization.

We notice first this theorem: *the quantities u_{ik} and v_{ik} are cogredient.*

This follows from equations (33), together with the corresponding equations for $X_\lambda(u_{ik})$, as well as from the following formulæ, which express the infinitesimal transformations of u_{ik} and v_{ik}. It will be found from (16) and (20), that

$$\begin{cases} \dfrac{\delta u_{11}}{\delta t} = \phi_{21}u_{12} - \phi_{12}u_{21}, \\ \dfrac{\delta u_{12}}{\delta t} = (\phi_{22} - \phi_{11})u_{12} + \phi_{12}(u_{11} - u_{22}), \\ \dfrac{\delta u_{21}}{\delta t} = (\phi_{11} - \phi_{22})u_{21} + \phi_{21}(u_{22} - u_{11}), \\ \dfrac{\delta u_{22}}{\delta t} = -\phi_{21}u_{12} + \phi_{12}u_{21}, \end{cases} \tag{37}$$

and from (16), (32), and (37),

$$\begin{cases} \dfrac{\delta v_{11}}{\delta t} = \phi_{21}v_{12} - \phi_{12}v_{21}, \\ \dfrac{\delta v_{12}}{\delta t} = (\phi_{22} - \phi_{11})v_{12} + \phi_{12}(v_{11} - v_{22}), \\ \dfrac{\delta v_{21}}{\delta t} = (\phi_{11} - \phi_{22})v_{21} + \phi_{21}(v_{22} - v_{11}), \\ \dfrac{\delta v_{22}}{\delta t} = -\phi_{21}v_{12} + \phi_{12}v_{21}. \end{cases} \tag{38}$$

Now certain combinations of the u_{ik}'s and p_{ik}'s, viz. $v_{11} + v_{22}$ and $v_{11}v_{22} - v_{12}v_{21}$, were seminvariants. Since the v_{ik}'s are cogredient with the u_{ik}'s, the same combinations with v_{ik} in place of u_{ik} will also be seminvariants.

Let us, therefore, put

$$(39)\qquad \begin{cases} w_{11} = 2v'_{11} + p_{12}v_{21} - p_{21}v_{12}, \\ w_{12} = 2v'_{12} + (p_{11} - p_{22})v_{12} - p_{12}(v_{11} - v_{22}), \\ w_{21} = 2v'_{21} - (p_{11} - p_{22})v_{21} + p_{21}(v_{11} - v_{22}), \\ w_{22} = 2v'_{22} - p_{12}v_{21} + p_{21}v_{12}; \end{cases}$$

then we know that $w_{11} + w_{22}$ and $w_{11}w_{22} - w_{12}w_{21}$ are seminvariants. But

$$w_{11} + w_{22} = 2(v'_{11} + v'_{22}) = 4I',$$

while

$$(40)\qquad L = w_{11}w_{22} - w_{12}w_{21}$$

is the new seminvariant. That it is independent of the other eight can be easily seen by considering the special case in which $p_{ik} = 0$.

We now have all the seminvariants, viz.: I, J, K, L, and their derivatives. For, suppose we wish to find the seminvariants involving $p^{(4)}_{ik}$ and $q^{(3)}_{ik}$. They are determined by a system of $24 - 1 = 23$ independent equations with 36 independent variables. Therefore, there exist $36 - 23 = 13$ such seminvariants. But they are merely the 8 seminvariants (36) and L, I''', J''', K'', L'.

Thus all seminvariants of the system (1) *for* $m = n = 2$, *are functions of the quantities* I, J, K, L, *and of their derivatives.*

It is interesting to note what would be the result of continuing our above process for obtaining seminvariants. *Suppose we had formed*

$$t_{11} = 2w'_{11} + p_{12}w_{21} - p_{21}w_{12}, \text{ etc.}$$

Then would

$$\begin{cases} t_{11} - t_{22} = g_1(u_{11} - u_{22}) + g_2(v_{11} - v_{22}) + g_3(w_{11} - w_{22}), \\ t_{12} = g_1u_{12} + g_2v_{12} + g_3w_{12}, \\ t_{21} = g_1u_{21} + g_2v_{21} + g_3w_{21}, \end{cases}$$

where g_1, g_2, g_3 *are seminvariants.*

For g_1, g_2, g_3 are the quotients of determinants of the third order formed out of the matrix

$$\begin{vmatrix} t_{11} - t_{22}, & u_{11} - u_{22}, & v_{11} - v_{22}, & w_{11} - w_{22} \\ t_{12}, & u_{12}, & v_{12}, & w_{12} \\ t_{21}, & u_{21}, & v_{21}, & w_{21} \end{vmatrix},$$

and an account of the cogrediency of these quantities, such quotients are seminvariants. In fact, the determinants themselves are seminvariants.

§ 4. *An application of the theory of seminvariants.*

The theory, developed in the preceding paragraphs, was first suggested by the following considerations. Let

$$D(y) = \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0 \tag{41}$$

be a homogeneous, linear, differential equation, and let $y_1, \cdots, y_n$ form a fundamental system of (41), so that

$$D(y_i) = 0 \qquad (i = 1, 2, \cdots, n). \tag{41a}$$

Suppose that the coefficients of (41) are uniform functions of x, and let $x = a_\mu$ be a singular point of one or all of these coefficients. If the variable x describes a closed circuit around this singular point a_μ, $y_1, \cdots, y_n$ will, in general, undergo a linear substitution with constant coefficients, changing into

$$\bar{y}_k = \sum_{i=1}^{n} \lambda_{ki}^{(\mu)} y_i \qquad (k = 1, 2, \cdots, n),$$

where the determinant $|\lambda_{ki}^{(\mu)}| \neq 0$. Denote this substitution by A_μ, so that we may write

$$y_k = A_\mu y_k. \tag{42}$$

Now let us put in (41a)

$$y_k = \sum_{i=1}^{n} a_{ki} \eta_i = S\eta_k \qquad (k = 1, 2, \cdots, n), \tag{43}$$

where again the determinant

$$|a_{ki}| \neq 0,$$

and where a_{ki} are uniform functions of x. Then $\eta_1, \cdots, \eta_n$ will verify a system of n linear differential equations, obtained from (41a) by the transformation (43). This system is not a general system but has the special property that corresponding to a circuit of x around a_μ, $\eta_1, \cdots, \eta_n$ undergo the substitution

$$S^{-1} A_\mu S,$$

which is the transformed of A_μ by the substitution S, and has as its coefficients uniform functions of x.

Conversely, if a system of n linear differential equations has this property, it is obviously possible to find a substitution

$$y_k = S\eta_k$$

which reduces it to the form (41a), or what amounts to the same thing, n linear

combinations of $\eta_1, \cdots, \eta_n$ can be formed with variable coefficients which form a fundamental system of a single linear differential equation of the nth order. Such functions as $\eta_1, \cdots, \eta_n$ are a very special kind of what I have called Λ functions.*

The question then arises: what are the necessary and sufficient conditions in order that a system of linear differential equations may have the property in question?

In our particular case, $m = n = 2$, it must then be possible to transform the given system into one of the form

$$y_i'' + py_i' + qy_i = 0 \qquad (i = 1, 2), \tag{44}$$

by a transformation of the form (43). The seminvariants of the given system must therefore be equal to those of (44). But for (44) we have

$$p_{12} = p_{21} = q_{12} = q_{21} = 0, \quad p_{11} = p_{22} = p, \quad q_{11} = q_{22} = q, \tag{45}$$

so that

$$\begin{cases} u_{11} = u_{22} = 2p' - 4q + p^2, & u_{12} = u_{21} = 0, \\ v_{11} = v_{22} = 2u_{11}', & v_{12} = v_{21} = 0, \\ w_{11} = w_{22} = 2v_{11}' = 4u_{11}'', & w_{12} = w_{21} = 0, \end{cases} \tag{46}$$

whence

$$I = 2u_{11}, \quad J = u_{11}^2, \quad K = 4(u_{11}')^2, \quad L = 16(u_{11}'')^2. \tag{47}$$

We have then in this case the relations:

$$I^2 - 4J = 0, \quad K - \left(\frac{dI}{dx}\right)^2 = 0, \quad L - 4\left(\frac{d^2I}{dx^2}\right)^2 = 0. \tag{48}$$

As will be seen from the expression deduced for the invariants in § 5, *all of the invariants vanish in this case.*

But the vanishing of all of the invariants, or even the fulfillment of the equations (48), while necessary, are not sufficient conditions for this case. The conditions (46) however, from which the others were derived, are both necessary and sufficient. It suffices to write down the system of invariant equations

$$u_{11} - u_{22} = 0, \quad u_{12} = u_{21} = 0, \tag{49}$$

for the other equations (46) follow from these.

That the conditions (49) are sufficient follows from the results of § 6. It will there be shown that every system of linear differential equations of the second order can by a transformation of the form

$$y_i = a_{i1}\eta_1 + a_{i2}\eta_2 \qquad (i = 1, 2), \tag{50}$$

*American Journal of Mathematics, vol. 16, No. 2, 1899.

be reduced to a form called the semi-canonical form. The result of this reduction in general is the system (82), which in our case reduces to

$$\eta_i'' = \tfrac{1}{4}u_{11}\eta_i \qquad (i=1,2), \tag{51}$$

i. e., η_1 and η_2 satisfy the same linear differential equation of the second order.

Thus we have seen that a system of the form

$$y_i'' + p_{i1}y_1' + p_{i2}y_2' + q_{i1}y_1 + q_{i2}y_2 = 0 \qquad (i=1,2) \tag{52}$$

can, by a transformation of form (50), *be reduced to the form* (44), *if and only if the conditions* (49) *are fulfilled. In that case, the integration of the system* (52) *is equivalent to the integration of a single linear differential equation of the second order. The invariants are all zero, and by a change of both dependent and independent variables this equation may be reduced to the form*

$$\frac{d^2u}{d\xi^2} = 0. \tag{53}$$

§ 5. *Calculation of the invariants for* $m = n = 2$.

The invariants of our system must obviously be functions of the seminvariants which we have already found. We shall therefore first investigate how these seminvariants are affected by a transformation of the independent variable x.

As before, let

$$y_i'' + p_{i1}y_1' + p_{i2}y_2' + q_{i1}y_1 + q_{i2}y_2 = 0 \qquad (i=1,2), \tag{54}$$

be the given system. If we introduce the new independent variable

$$\xi = \xi(x),$$

this system becomes

$$y_i'' + \pi_{i1}y_1' + \pi_{i2}y_2' + \kappa_{i1}y_1 + \kappa_{i2}y_2 = 0 \qquad (i=1,2), \tag{55}$$

where

$$\left\{\begin{aligned} &\pi_{11} = \frac{p_{11}}{\xi'} + \frac{\xi''}{(\xi')^2}, \quad \pi_{12} = \frac{p_{12}}{\xi'}, \quad \pi_{21} = \frac{p_{21}}{\xi'}, \quad \pi_{22} = \frac{p_{22}}{\xi'} + \frac{\xi''}{(\xi')^2}, \\ &\kappa_{ik} = \frac{q_{ik}}{(\xi')^2}. \end{aligned}\right. \tag{56}$$

For an infinitesimal transformation, we put

$$\xi(x) = x + \phi(x)\delta t, \quad \delta x = \phi(x)\delta t, \tag{57}$$

where δt is an infinitesimal and $\phi(x)$ an arbitrary function. We shall then have

$$\left\{\begin{aligned} &\delta p_{11} = (-\phi' p_{11} + \phi'')\delta t, \quad \delta p_{12} = -\phi' p_{12}\delta t, \\ &\delta p_{21} = -\phi' p_{21}\delta t, \qquad \delta p_{22} = (-\phi' p_{22} + \phi'')\delta t, \\ &\delta q_{ik} = -2\phi' q_{ik}\delta t. \end{aligned}\right. \tag{58}$$

If f is any function of x, and f' its derivative, we have

$$\delta f' = \delta\frac{df}{dx} = \frac{\delta(df)}{dx} - \frac{df}{dx^2}\delta(dx) = \frac{d}{dx}(\delta f) - f'\frac{d}{dx}(\delta x)$$

or

$$\delta f' = \frac{d}{dx}(\delta f) - f'\phi'(x)\delta t. \tag{59}$$

Thus we find from (58)

$$\begin{cases} \delta p'_{11} = (\phi^{(3)} - \phi'' p_{11} - 2\phi' p'_{11})\delta t, \\ \delta p'_{12} = \quad (-\phi'' p_{12} - 2\phi' p'_{12})\delta t, \\ \delta p'_{21} = \quad (-\phi'' p_{21} - 2\phi' p'_{21})\delta t, \\ \delta p'_{22} = (\phi^{(3)} - \phi'' p_{22} - 2\phi' p'_{22})\delta t. \end{cases} \tag{60}$$

Therefore, making use of (20), (58), and (60), we find

$$\begin{cases} \delta u_{11} = (2\phi^{(3)} - 2\phi' u_{11})\delta t, \quad \delta u_{12} = -2\phi' u_{12}\delta t, \\ \delta u_{21} = -2\phi' u_{21}\delta t \qquad , \quad \delta u_{22} = (2\phi^{(3)} - 2\phi' u_{22})\delta t. \end{cases} \tag{61}$$

Consequently we shall have, remembering the definition of I and J,

$$\begin{cases} \delta I = (4\phi^{(3)} - 2\phi' I)\delta t, \\ \delta J = (2\phi^{(3)} I - 4\phi' J)\delta t. \end{cases} \tag{62}$$

In the same way we find

$$\begin{cases} \delta v_{11} = (4\phi^{(4)} - 4\phi'' u_{11} - 3\phi' v_{11})\delta t, \\ \delta v_{12} = (-4\phi'' u_{12} - 3\phi' v_{12})\delta t, \\ \delta v_{21} = (-4\phi'' u_{21} - 3\phi' v_{21})\delta t, \\ \delta v_{22} = (4\phi^{(4)} - 4\phi'' u_{22} - 3\phi' v_{22})\delta t, \end{cases} \tag{63}$$

whence

$$\delta K = (8\phi^{(4)} I' - 8\phi'' J' - 6\phi' K)\delta t, \tag{64}$$

after a slight transformation, involving equation (67) for J'.

We find further

$$\begin{cases} \delta w_{11} = (8\phi^{(5)} - 8\phi^{(3)} u_{11} - 10\phi'' v_{11} - 4\phi' w_{11})\delta t, \\ \delta w_{12} = (-8\phi^{(3)} u_{12} - 10\phi'' v_{12} - 4\phi' w_{12})\delta t, \\ \delta w_{21} = (-8\phi^{(3)} u_{21} - 10\phi'' v_{21} - 4\phi' w_{21})\delta t, \\ \delta w_{22} = (8\phi^{(5)} - 8\phi^{(3)} u_{22} - 10\phi'' v_{22} - 4\phi' w_{22})\delta t; \end{cases} \tag{65}$$

whence follows

$$\delta L = [32\phi^{(5)} I'' - 16\phi^{(3)}(2J'' - K') - 20\phi'' K' - 8\phi' L]\delta t\,, \tag{66}$$

if it be noted that

$$\begin{cases} u_{11}v_{22} + u_{22}v_{11} - u_{12}v_{21} - u_{21}v_{12} = 2J'\,, \\ v_{11}w_{22} + v_{22}w_{11} - v_{12}w_{21} - v_{21}w_{12} = 2K'\,, \\ w_{11}u_{22} + w_{22}u_{11} - w_{12}u_{21} - w_{21}u_{12} = 2(2J'' - K')\,. \end{cases} \tag{67}$$

Let us now, by an obvious extension of the theory of invariants of a single linear differential equation, assign to p_{ik} the weight -1, to q_{ik} the weight -2, while to $p_{ik}^{(\lambda)}$ and $q_{ik}^{(\mu)}$ are assigned the weights $-1-\lambda$, $-2-\mu$ respectively. Then it is easy to see that the following statements are true.

1. *Every absolute invariant is isobaric in the coefficients and of weight zero.*

2. *If an absolute invariant is a rational function of the coefficients and of their derivatives, it is the quotient of two relative invariants of the same weight.*

3. *A relative invariant is isobaric in the coefficients, and if the common weight of all of its terms is $-\nu$, it satisfies the equation*

$$\xi(x)^{\nu}\theta_{\nu}(\xi) = \theta_{\nu}(x)\,, \tag{68}$$

or, for infinitesimal transformations,

$$\delta\theta_{\nu} = -\nu\phi'(x)\theta_{\nu}\delta t\,. \tag{68a}$$

We shall speak of such an invariant as being of weight ν, rather than $-\nu$. The negative weights have been introduced, following the example of FORSYTH, principally because they have a decided advantage when we consider *covariants*.

The proof of the above three statements is essentially the same as in the case of a single linear differential equation, and need not, therefore, be given.

It is now a simple matter to find the rational invariants of any assigned weight, or else to establish their non-existence.

First, it is clear that no such invariants of weights 1, 2, 3 exist. An invariant of weight 4 must verify the equation

$$\delta\theta_4 = -4\phi'(x)\theta_4\delta t\,.$$

The invariant is

$$\theta_4 = I^2 - 4J. \tag{69}$$

An invariant of weight 5 must be of the form

$$aII' + bI^{(3)} + cJ'.$$

We find that no such invariant exists, or that it vanishes identically.

An invariant of weight 6 must verify the equation

$$\frac{\delta\theta_6}{\delta t} = -6\phi'\theta_6.$$

The most general expression of weight 6 is

$$\theta_6 = aI^3 + bIJ + cK + dI^{(4)} + eJ'' + fII'' + gI'^2.$$

We find

$$\begin{aligned}\frac{\delta\theta_6}{\delta t} = {} & 3aI^2(4\phi^{(3)} - 2\phi' I) + bI(2\phi^{(3)}I - 4\phi' J) + bJ(4\phi^{(3)} - 2\phi' I) \\ & + c(8\phi^{(4)}I - 8\phi'' J' - 6\phi' K) \\ & + d(4\phi^{(7)} - 2\phi^{(5)}I - 9\phi^{(4)}I' - 12\phi^{(3)}I'' - 10\phi'' I^{(3)} - 6\phi' I^{(4)}) \\ & + e(2\phi^{(5)}I + 4\phi^{(4)}I' + 2\phi^{(3)}I'' - 4\phi^{(3)}J - 9\phi'' J' - 6\phi' J'') \\ & + fI(4\phi^{(5)} - 2\phi^{(3)}I - 5\phi'' I' - 4\phi' I'') + fI''(4\phi^{(3)} - 2\phi' I) \\ & + 2gI'(4\phi^{(4)} - 2\phi'' I - 3\phi' I'),\end{aligned}$$

and this must be equal to $-6\phi'\theta_6$ for all values of $\phi, \phi', \cdots, I, I', \cdots, J, J', \cdots, K$. We find therefore the equations:

$$\begin{cases} d = 0, \quad e + 2f = 0, \quad 2c + e + 2g = 0, \quad 6a + b - f = 0, \\ b - e = 0, \quad 8c + 9e = 0, \quad 5f + 4g = 0, \end{cases}$$

whence

$$a = -\tfrac{1}{4}e, \quad b = e, \quad c = -\tfrac{9}{8}e, \quad d = 0, \quad f = -\tfrac{1}{2}e, \quad g = \tfrac{5}{8}e.$$

Putting $e = -8$, we find

$$\theta_6 = 2I(I^2 - 4J) + 5(K - I'^2) + 4(K - 2J'' + II''). \tag{70}$$

There is no invariant of weight 7, and there are two independent invariants of weight 8, one of which is θ_4^2, while the other is

$$\begin{aligned}\theta_8 = {} & 143(L - 4I''^2) - 54(I^2 + 4J)\theta_4 - 20I''\theta_4 + 25I'\theta_4' - 206I\theta_4'' \\ & - 20\theta_4^{(4)} - 902I(K - I'^2) - 220(K'' - 2I'I^{(3)} - 2I''^2).\end{aligned} \tag{71}$$

We can easily find an invariant of weight 10, without going through this general process. We have

$$\frac{\delta(K - I'^2)}{\delta t} = 4\phi''(II' - 2J') - 6\phi'(K - I'^2),$$

$$\frac{\delta(II' - 2J')}{\delta t} = -2\phi''(I^2 - 4J) - 5\phi'(II' - 2J'),$$

whence eliminating ϕ'', we find

$$\frac{\delta\theta_{10}}{\delta t} = -10\phi'\theta_{10},$$

where

$$\theta_{10} = (I^2 - 4J)(K - II') + (II' - 2J')^2. \tag{72}$$

From any two invariants θ_λ and θ_μ, of weights λ and μ respectively, we can always deduce a third. For we have

$$\frac{\theta_\lambda^\mu(\xi)}{\theta_\mu^\lambda(\xi)} = \frac{\theta_\lambda^\mu(x)}{\theta_\mu^\lambda(x)}$$

an absolute invariant. Hence, by logarithmic differentiation,

$$\left\{\mu\frac{\theta'_\lambda(\xi)}{\theta_\lambda(\xi)} - \lambda\frac{\theta'_\mu(\xi)}{\theta_\mu(\xi)}\right\}\xi' = \mu\frac{\theta'_\lambda(x)}{\theta_\lambda(x)} - \lambda\frac{\theta'_\mu(x)}{\theta_\mu(x)}.$$

Therefore

$$\theta_{\lambda+\mu+1} = \mu\theta_\mu\theta'_\lambda - \lambda\theta_\lambda\theta'_\mu \tag{73}$$

is a new invariant of weight $\lambda + \mu + 1$. It is called by FORSYTH the *Jacobian* of θ_λ and θ_μ, in his theory of invariants of a single equation.

We thus obtain

$$\left\{\begin{aligned}
\theta_{11} &= 3\theta_6\theta'_4 - 2\theta_4\theta'_6, \\
\theta_{13} &= 2\theta_8\theta'_4 - \theta_4\theta'_8, \\
\theta_{15} &= 5\theta_{10}\theta'_4 - 2\theta_4\theta'_{10}, \\
\vartheta_{15} &= 4\theta_8\theta'_6 - 3\theta_6\theta'_8, \\
\theta_{17} &= 5\theta_{10}\theta'_6 - 3\theta_6\theta'_{10}, \\
\theta_{19} &= 5\theta_{10}\theta'_8 - 4\theta_8\theta'_{10},
\end{aligned}\right. \tag{74}$$

from which still others can be derived. Of all of the invariants found so far

$$\theta_4,\ \theta_6,\ \theta_{10},\ \theta_{15}$$

are the only ones which involve no higher derivatives of p_{ik} than the third, and no higher derivatives of q_{ik} than the second, or what amounts to the same thing, these are the only invariants found so far which depend only on the seminvariants

$$I,\ I',\ I'';\ J,\ J',\ J'';\ K,\ K';\ L. \tag{75}$$

Moreover the invariants (74) are not all independent. For instance we have the syzygy

$$2\theta_4\theta_{17} + 5\theta_{10}\theta_{11} - 6\theta_6\theta_{15} = 0\,. \tag{76}$$

In order to obtain *all* of the invariants depending only upon the seminvariants (75), we write down the partial differential equations which such invariants must verify.

From the expressions for δI, $\delta I'$, $\delta I''$. δJ, etc., δL, we find that these partial differential equations are five in number, viz.:

$$\left\{\begin{aligned}
Y_1 f &= 2I\frac{\partial f}{\partial I} + 3I'\frac{\partial f}{\partial I'} + 4I''\frac{\partial f}{\partial I''} + 4J\frac{\partial f}{\partial J} + 5J'\frac{\partial f}{\partial J'} + 6J''\frac{\partial f}{\partial J''} \\
&\qquad + 6K\frac{\partial f}{\partial K} + 7K'\frac{\partial f}{\partial K'} + 8L\frac{\partial f}{\partial L} = 0\,, \\
Y_2 f &= -2I\frac{\partial f}{\partial I'} - 5I'\frac{\partial f}{\partial I''} - 4J\frac{\partial f}{\partial J'} - 9J'\frac{\partial f}{\partial J''} - 8J'\frac{\partial f}{\partial K} \\
&\qquad - (8J'' + 6K)\frac{\partial f}{\partial K'} - 20K'\frac{\partial f}{\partial L} = 0\,, \\
Y_3 f &= 4\frac{\partial f}{\partial I} - 2I\frac{\partial f}{\partial I''} + 2I\frac{\partial f}{\partial J} + 2I'\frac{\partial f}{\partial J'} + 2(I'' - 2J)\frac{\partial f}{\partial J''} \\
&\qquad - 8J'\frac{\partial f}{\partial K'} - 16(2J'' - K)\frac{\partial f}{\partial L} = 0\,, \\
Y_4 f &= 4\frac{\partial f}{\partial I'} + 2I\frac{\partial f}{\partial J'} + 4I'\frac{\partial f}{\partial J''} + 8I'\frac{\partial f}{\partial K} + 8I''\frac{\partial f}{\partial K'} = 0\,, \\
Y_5 f &= 4\frac{\partial f}{\partial I''} + 2I\frac{\partial f}{\partial J''} + 8I'\frac{\partial f}{\partial K'} + 32I''\frac{\partial f}{\partial L} = 0\,,
\end{aligned}\right. \tag{77}$$

where $Y_i f = 0$ is obtained by equating to zero the coefficient of $\phi^{(i)}$ in δf.

They are independent and, therefore, have $9 - 5 = 4$ independent solutions, i. e., there are four independent absolute, or five independent relative invariants. We have already found four of these, viz.: θ_4, θ_6, θ_{10}, θ_{15}. The fifth invariant, found by integrating (77), is

$$\theta_{22} = \theta_{10}[(I^2 - 4J)(L - 4I''^2) + 4(II'' - 2J'' + K)^2] - \theta_{10}'^2\,.$$

But this can be shown to be divisible by θ_4, so that we may complete our system of invariants by taking θ_{22}/θ_4, which is

$$\begin{aligned}
\theta_{15} = \theta_{10}(L - 4I''^2) + 4(K - I'^2)(II'' - 2J'' + K)^2 - \theta_4(K' - 2I'I'')^2 \\
- 2\theta_4'(K' - 2I'I'')(II'' - 2J'' + K).
\end{aligned} \tag{78}$$

§ 6. *Canonical forms of a system of two linear differential equations of the second order.*

Our system of equations (54) can always be transformed into another which contains no first derivatives, by a transformation of the form

$$y_i = a_{i1}\eta_1 + a_{i2}\eta_2 \qquad (i=1,2),$$

where a_{ik} are functions of x. For, on making this transformation, we find

$$(79)\quad \begin{cases} a_{i1}\eta_1'' + a_{i2}\eta_2'' + (2a_{i1}' + p_{i1}a_{11} + p_{i2}a_{21})\eta_1' + (2a_{i2}' + p_{i1}a_{12} + p_{i2}a_{22})\eta_2' \\ \qquad + (a_{i1}'' + p_{i1}a_{11}' + p_{i2}a_{21}' + q_{i1}a_{11} + q_{i2}a_{21})\eta_1 \\ \qquad + (a_{i2}'' + p_{i1}a_{12}' + p_{i2}a_{22}' + q_{i1}a_{12} + q_{i2}a_{22})\eta_2 = 0 \end{cases} \qquad (i=1,2),$$

so that, if we take a_{ik} subject to the conditions

$$(80)\quad \begin{cases} a_{11}' = -\tfrac{1}{2}(p_{11}a_{11} + p_{12}a_{21}), \quad a_{12}' = -\tfrac{1}{2}(p_{11}a_{12} + p_{12}a_{22}), \\ a_{21}' = -\tfrac{1}{2}(p_{21}a_{11} + p_{22}a_{21}), \quad a_{22}' = -\tfrac{1}{2}(p_{21}a_{12} + p_{22}a_{22}), \end{cases}$$

equations (79) will contain no terms in η_1' and η_2'. The functions (a_{11}, a_{21}) and (a_{12}, a_{22}) must therefore be taken as solutions of the same system of linear differential equations, viz.,

$$(81)\quad \begin{cases} \alpha' = -\tfrac{1}{2}(p_{11}\alpha + p_{12}\beta), \\ \beta' = -\tfrac{1}{2}(p_{21}\alpha + p_{22}\beta). \end{cases}$$

Moreover (a_{11}, a_{21}) and (a_{12}, a_{22}) must be independent solutions of (81), for otherwise the determinant

$$a_{11}a_{22} - a_{12}a_{21}$$

would vanish. The functions a_{ik} can therefore be determined so as to effect the desired result, by integrating (81).

If one makes use of (80) and the equations obtained from (80) by differentiation the transformed system (79) becomes

$$(82)\quad \begin{cases} a_{11}\eta_1'' + a_{12}\eta_2'' = \tfrac{1}{4}(a_{11}u_{11} + a_{21}u_{12})\eta_1 + \tfrac{1}{4}(a_{12}u_{11} + a_{22}u_{12})\eta_2, \\ a_{21}\eta_1'' + a_{22}\eta_2'' = \tfrac{1}{4}(a_{11}u_{21} + a_{21}u_{22})\eta_1 + \tfrac{1}{4}(a_{12}u_{21} + a_{22}u_{22})\eta_2, \end{cases}$$

where the quantities u_{ik} have been previously defined.

Thus every binary system of homogeneous linear differential equations of the second order can be converted into another, involving no first derivatives, i. e., *into one for which* $p_{ik} = 0$. We will say that this transformed system has the *semi-canonical* form.

Suppose we have reduced a system to its semi-canonical form. *Let us find the most general sub-group* G' *of our general infinite continuous group* G, *which leaves the semi-canonical form of the system unaltered.*

In the semi-canonical form $p_{ik} = 0$. Let us make the most general infinitesimal transformation of the general group G. The coefficients of the first derivatives in the transformed system will be, according to (16) and (58),

$$\begin{cases} \delta p_{11} = (2\phi'_{11} - \phi'')\delta t, & \delta p_{12} = 2\phi'_{12}\delta t, \\ \delta p_{21} = 2\phi'_{21}\delta t, & \delta p_{22} = (2\phi'_{22} - \phi'')\delta t, \end{cases}$$

for the infinitesimal transformations of the coefficients due to infinitesimal transformations of both dependent and independent variables are equal to those due to the transformations of the dependent variables alone plus those due to the transformation of the independent variable alone.*

But for all of the transformations of our sub-group G', these quantities must vanish. Therefore must

$$\phi'_{12} = \phi'_{21} = 0, \quad 2\phi'_{11} - \phi'' = 0, \quad 2\phi'_{22} - \phi'' = 0,$$

or, if we denote by c_{ik} arbitrary constants,

$$(83) \qquad \begin{cases} \phi_{11} = c_{11} + \tfrac{1}{2}\phi', & \phi_{12} = c_{12}, \\ \phi_{21} = c_{21}, & \phi_{22} = c_{22} + \tfrac{1}{2}\phi', \end{cases}$$

so that the sub-group G' depends upon 4 arbitrary parameters c_{ik}, and an arbitrary function $\phi'(x)$.

We can determine *a sub-group G'' of the subgroup G', namely, that which leaves $q_{11} + q_{22}$ invariant.*

We have, in the same way as above, according to (16) and (58),

$$\delta(q_{11} + q_{22}) = [2\phi'(q_{11} + q_{22}) + \phi^{(2)}_{11} + \phi^{(2)}_{22}]\delta t,$$

so that for this sub-group we have the further relation

$$(84) \qquad \phi^{(2)}_{11} + \phi^{(2)}_{22} = -2\phi'(q_{11} + q_{22}),$$

or, using (83),

$$(85) \qquad \phi^{(2)}_{11} + \phi^{(2)}_{22} + 2[c_{11} + c_{22} - (\phi_{11} + \phi_{22})](q_{11} + q_{22}) = 0,$$

a linear differential equation of the second order for $\phi_{11} + \phi_{22}$. Its integration introduces two arbitrary constants; then ϕ' is found from (84), and the quadrature

$$\phi = \int \phi' dx$$

gives rise to a third constant. *Thus G'' is a seven-parameter group.* We can also represent G'' by (83), together with

$$(86) \qquad -\phi^{(3)} = 2\phi'(q_{11} + q_{22}),$$

* We have changed the sign of $\varphi(x)$, so as to make the infinitesimal transformation of the independent variable harmonize with those of the dependent variables.

an equation, whose integration gives rise to three constants. These together with the four constants c_{ik} are the parameters of the group.

The semi-canonical form of the system can be further simplified by the reduction to what we shall call the *canonical* form.

Suppose we have reduced our system by a first transformation to the semi-canonical form

$$y_i'' + q_{i1}y_1 + q_{i2}y_2 = 0 \qquad (i=1,2). \tag{87}$$

If now we make the transformation

$$\xi = \xi(x), \quad y_i = \beta_{i1}\eta_1 + \beta_{i2}\eta_2 \qquad (i=1,2),$$

we find

$$\begin{aligned}\beta_{i1}\xi'^2\frac{d^2\eta_1}{d\xi^2} + \beta_{i2}\xi'^2\frac{d^2\eta_2}{d\xi^2} + (\beta_{i1}\xi'' + 2\beta_{i1}'\xi')\frac{d\eta_1}{d\xi} + (\beta_{i2}\xi'' + 2\beta_{i2}'\xi')\frac{d\eta_2}{d\xi} \\ + (\beta_{i1}'' + q_{i1}\beta_{11} + q_{i2}\beta_{21})\eta_1 + (\beta_{i2}'' + q_{i1}\beta_{12} + q_{i2}\beta_{22})\eta_2 = 0 \qquad (i=1,2),\end{aligned} \tag{88}$$

which is again in the semi-canonical form if

$$\beta_{ik}\xi'' + 2\beta_{ik}'\xi' = 0,$$

or

$$\beta_{ik} = \frac{e_{ik}}{\sqrt{\xi'}},$$

where e_{ik} are arbitrary constants, whose determinant does not vanish. Put

$$e_{11} = e_{22} = 1, \quad e_{12} = e_{21} = 0,$$

or

$$\beta_{11} = \beta_{22} = \beta = \frac{1}{\sqrt{\xi'}}, \quad \beta_{12} = \beta_{21} = 0. \tag{89}$$

Then (88) becomes

$$\begin{cases}\beta\xi'^2\dfrac{d^2\eta_1}{d\xi^2} + (\beta'' + q_{11}\beta)\eta_1 + q_{12}\beta\eta_2 = 0, \\ \beta\xi'^2\dfrac{d^2\eta_2}{d\xi^2} + q_{21}\beta\eta_1 + (\beta'' + q_{22}\beta)\eta_2 = 0,\end{cases} \tag{90}$$

or

$$\frac{d^2\eta_i}{d\xi^2} + \rho_{i1}\eta_1 + \rho_{i2}\eta_2 = 0 \qquad (i=1,2). \tag{91}$$

Now β can be determined that in (91), $\rho_{11} + \rho_{22} = 0$; for this purpose it is only necessary to take for β a solution of the linear differential equation

$$2\beta'' + (q_{11} + q_{22})\beta = 0. \tag{92}$$

We have proved the following theorem:

Every system of linear, homogeneous differential equations of the second order can be converted into a system having the canonical form

$$\frac{d^2\eta_i}{d\xi^2} + \rho_{i1}\eta_1 + \rho_{i2}\eta_2 \qquad (i=1,2),$$

where

$$\rho_{11} + \rho_{22} = 0\,.$$

In order to effect this reduction, it is necessary to integrate a system of homogeneous linear differential equations of the first order (81), *a single homogeneous linear differential equation of the second order* (92), *and finally to effect the quadrature*

$$\xi(x) = \int \frac{dx}{\beta^2}\,.$$

This canonical form of the system corresponds to the LAGUERRE-FORSYTH form of a single linear differential equation.

The sub-group of G which leaves this canonical form unaltered is especially simple. It is the group G''' for the particular case that $q_{11} + q_{22} = 0$, whence, according to (86),

$$\phi^{(3)} = 0\,, \quad \text{or} \quad \phi = \lambda + \mu x + \nu x^2,$$

and from (83),

$$\begin{cases} \phi_{11} = c_{11} + \frac{1}{2}\mu + \nu x\,, & \phi_{12} = c_{12}\,, \\ \phi_{21} = c_{21}\,, & \phi_{22} = c_{22} + \frac{1}{2}\mu + \nu x\,. \end{cases}$$

The finite transformations of this group are

$$(93) \qquad \xi = \frac{ax+\beta}{\gamma x+\delta}\,, \quad \eta_i = \frac{C_{i1}y_1 + C_{i2}y_2}{\gamma x+\delta} \qquad (i=1,2)\,.$$

The functions called, by Forsyth, *quadriderivatives* are invariants for this sub-group. If θ_σ is an invariant of weight σ, then

$$(94) \qquad \theta_{\sigma,1} = 2\sigma\theta_\sigma\theta''_\sigma - (2\sigma+1)\theta'^2_\sigma$$

is what FORSYTH calls the quadriderivative of θ_σ.

We find that for the general infinitesimal transformations of the general group G,

$$\delta\theta_{\sigma,1} = -2\sigma^2\theta^2_\sigma\phi^{(3)}\delta t - 2(\sigma+1)\phi'\theta_{\sigma,1}\delta t\,.$$

For our sub-group $\phi^{(3)} = 0$, so that $\theta_{\sigma,1}$ is indeed an invariant of the sub-group. Its weight is $2\sigma + 2$.

§ 7. *Covariants.*

We shall not treat of covariants in an exhaustive manner in this paper. It will be sufficient to note a few simple theorems.

First we may observe that *no absolute covariant exists which depends only upon* y_1, y_2, dy_1/dx, dy_2/dx, *etc., i. e., which does not contain* x *explicitly, and makes no use of the fact that* y_1 *and* y_2 *are solutions of* (54).

For we have, taking infinitesimal transformations of the dependent variables only,

$$\begin{cases} \delta y_i = (\phi_{i1} y_1 + \phi_{i2} y_2)\delta t\,, \\ \delta y_i' = (\phi_{i1} y_1' + \phi_{i2} y_2' + \phi_{i1}' y_1 + \phi_{i2}' y_2)\delta t\,, \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ \delta y_i^{(m-1)} = (\phi_{i1} y_1^{(m-1)} + \phi_{i2} y_2^{(m-1)} + \cdots + \phi_{i2}^{(m-1)} y_2)\delta t\,. \end{cases}$$

But this group in $2m$ variables is always transitive. For if we take the special case obtained by putting ϕ_{ik}, ϕ_{ik}', ϕ_{ik}'', etc., equal to arbitrary constants, the group becomes a finite $4m$ parameter group in $2m$ variables which is a particular case of a general group of linear substitutions with mn^2 parameters and mn variables. This general group has been studied by the author and was found to be transitive.* Therefore the more general infinite group is also transitive, i. e., it has no invariants, and thus the theorem is proved.

It is even impossible to construct functions z_1 *and* z_2 *of* y_1, y_2 *and of their successive derivatives, not involving* x *explicitly, which shall be cogredient with* y_1 *and* y_2.

For, if it were not so, from z_1 and z_2 could then be constructed a third cogredient set, say η_1 and η_2, and the quotient

$$\frac{y_1 z_2 - y_2 z_1}{y_1 \eta_2 - y_2 \eta_1}$$

would be an absolute covariant of the form whose non-existence we have just shown.

An infinite number of covariants, containing x explicitly, can be constructed as follows. Put

$$\begin{cases} \bar{y}_i = y_i'' + p_{i1} y_1' + p_{i2} y_2' + q_{i1} y_1 + q_{i2} y_2, \\ \bar{y}_i = \bar{y}_i'' + p_{i1} \bar{y}_1' + p_{i2} \bar{y}_2' + q_{i1} \bar{y}_1 + q_{i2} \bar{y}_2, \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Then $\bar{y}_i$, $\bar{\bar{y}}_i$, etc., are cogredient with y_i. Therefore the determinants

$$y_1 \bar{y}_2 - y_2 \bar{y}_1, \quad y_1 \bar{\bar{y}}_2 - y_2 \bar{\bar{y}}_1, \text{ etc.,}$$

*Proceedings of the California Academy of Sciences, vol. 1, no. 6.

are covariants. All of these vanish in consequence of the given system of differential equations.

Their covariant property is expressed by an equation of the form

$$C(\eta_1, \eta_2, \xi) = \frac{a_{11}a_{22} - a_{12}a_{21}}{\xi'(x)^2} C(y_1, y_2, x),$$

where

$$\eta_i = a_{i1}y_1 + a_{i2}y_2, \quad \xi = \xi(x) \qquad (i = 1, 2),$$

is the transformation of the variables.

Suppose we have reduced our given system to its canonical form. Then

$$y_1y_2' - y_2y_1'$$

is a covariant of the sub-group (93) which leaves this canonical form unchanged.

To find all of the covariants of the system we could proceed as in the case of the invariants by setting up the system of partial differential equations which they satisfy. We might also construct functions cogredient with y_1 and y_2. The determinant of two such systems would be a covariant.

UNIVERSITY OF CALIFORNIA, BERKELEY,
October 6, 1900.

DIVERGENT AND CONDITIONALLY CONVERGENT SERIES WHOSE PRODUCT IS ABSOLUTELY CONVERGENT*

BY

FLORIAN CAJORI

§1. *Introduction.*

It has been shown by ABEL that, if the product:

$$\sum_{n=0}^{n=\infty}(u_0v_n + u_1v_{n-1} + \cdots + u_nv_0),$$

of two conditionally convergent series:

$$\sum_{n=0}^{n=\infty}u_n, \qquad \sum_{n=0}^{n=\infty}v_n,$$

is convergent, it converges to the product of their sums. Tests of the convergence of the product of conditionally convergent series have been worked out by A. PRINGSHEIM,† A. VOSS,‡ and myself.§ There exist certain conditionally-convergent series which yield a convergent result when they are raised to a certain positive integral power, but which yield a divergent result when they are raised to a higher power. Thus,

$$\sum_{n=1}^{n=\infty}(-1)^{n+1}\frac{1}{n^r},$$

where $r = 7/9$, is a conditionally convergent series whose fourth power is convergent, but whose fifth power is divergent.‖ These instances of conditionally

* Presented to the Society April 28, 1900. Received for publication April 28, 1900.

† Mathematische Annalen, vol. 21 (1883), p. 327; vol. 26 (1886), p. 157.

‡ Mathematische Annalen, vol. 24 (1884), p. 42.

§ American Journal of Mathematics, vol. 15 (1893), p. 339; vol. 18 (1896), p. 195; Bulletin of the American Mathematical Society, (2) vol. 1 (1895), p. 180.

‖ This may be a convenient place to point out a slight and obvious extension of the results which I have published in the American Journal of Mathematics, vol. 18, p. 201. It was proved there that the conditionally convergent series:

$$\sum_{n=1}^{n=\infty}(-1)^{n+1}\frac{1}{n^r}, \qquad (0 < r \leqq 1),$$

when raised by Cauchy's multiplication rule to a positive integral power q, is convergent whenever $(q-1)/q < r$; but the power of the series is divergent, if $(q-1)/q > r$. The case for which $(q-1)/q = r$ was left untouched, but an examination of the formulæ appearing in that article readily yields the result that in this case the series is divergent. Hence the condition for divergency should be written $(q-1)/q \geqq r$.

convergent series, yielding powers of the series which converge less rapidly the higher the power and which for a sufficiently high power yield a divergent result, suggest the following questions: *Does the product of two conditionally convergent series never converge more rapidly than do one or both of the factor-series? Can the product of two conditionally convergent series or of a conditionally convergent and a divergent series in no case be absolutely convergent?*

The first doubt of the correctness of a negative reply arose in connection with the conditionally convergent series:

$$\sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n}.$$

This yields convergent results, no matter to how high a positive integral power it may be raised.* If the series is squared, the constituents, $1/(n-r)(r+1)$, which go to make up the nth term of the product, namely,

$$\sum_{r=0}^{r=n-1} \frac{1}{n-r} \frac{1}{r+1},$$

are all positive or all negative for that term. Hence the numerical value of the nth term is the sum of the numerical values of all the products $1/(n-r)(r+1)$. If the square of the series is indicated by

$$\sum_{n=1}^{n=\infty} a_n,$$

then all the constituents, $a_{n-r} \cdot 1/(r+1)$, which enter into the composition of the nth term in the cube of the series have all like signs in that term. Similarly for higher powers. Now it has been shown that all the positive integral powers of the conditionally convergent series:

$$\sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n}$$

continue to converge, even though each term of a given power of the series is numerically the sum of the numerical values of its constituents, and it is evident that, if the constituents of each term were partly positive and partly negative, each term would be numerically smaller. *Can the signs of the terms of two conditionally convergent series and the numerical values of its terms be so chosen that the product of the two series is absolutely convergent?*

In an important article on the multiplication of conditionally convergent series† A. PRINGSHEIM states that he knows no *a priori* reason why there might not be special cases in which such a product is absolutely convergent. "Aller-

* American Journal of Mathematics, vol. 18, p. 204.

† Mathematische Annalen, vol. 21, p. 332.

dings," he continues, "war es mir bisher nicht möglich an irgend einem bestimmten derartigen Falle die unbedingte Convergenz zu constatiren, und es dürfte dies — wenn es überhaupt solche Fälle giebt, was ja immerhin als fraglich erscheinen muss — wegen der zusammengesetzten Beschaffenheit der Reihenglieder w_ν eben nur sehr schwer gelingen."

§2. *Two typical examples.*

We have succeeded in devising special cases in which an absolutely convergent series is obtained as the result of multiplying two conditionally convergent series together, or one conditionally convergent series by a divergent series. The series obtained by removing the parentheses from the series:

$$S_1 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{4p+1}-\frac{1}{4p+4}+\frac{1}{4p+1}-\frac{1}{4p+4}\right),$$

$$S_2 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{4p+1}+\frac{1}{4p+4}-\frac{1}{4p+1}-\frac{1}{4p+4}\right),$$

are both conditionally convergent. Let the four terms obtained in either series, or the terms in a similar series, for any one positive integral value of p, be called a "set" of terms. The $4n$-th term of the product of S_1 and S_2 is

$$\sum_{p=0}^{p=n-1}\left(-\frac{1}{4n-4p}\,\frac{1}{4p+1}+\frac{1}{4n-4p-3}\,\frac{1}{4p+4}\right.$$
$$\left.+\frac{1}{4n-4p}\,\frac{1}{4p+1}-\frac{1}{4n-4p-3}\,\frac{1}{4p+4}\right).$$

We see that the sum of each set of constituents for the $4n$-th term is identically zero; therefore the entire $4n$-th term is identically zero.

The $(4n+1)$-th term of the product series is as follows:

$$\sum_{p=0}^{p=n'}\left(\frac{1}{4n-4p+1}\,\frac{1}{4p+1}-\frac{1}{4n-4p}\,\frac{1}{4p+4}\right.$$
$$\left.-\frac{1}{4n-4p-3}\,\frac{1}{4p+1}+\frac{1}{4n-4p}\,\frac{1}{4p+4}\right)$$
$$+\sum_{p=0}^{p=n'}\left(\frac{1}{4n-4p+1}\,\frac{1}{4p+1}+\frac{1}{4n-4p}\,\frac{1}{4p+4}\right.$$
$$\left.-\frac{1}{4n-4p-3}\,\frac{1}{4p+1}-\frac{1}{4n-4p}\,\frac{1}{4p+4}\right)-a,$$

where $n'=\frac{1}{2}n-1$ or $\frac{1}{2}(n-1)$, according as n is even or odd. When n is odd,

$$a=\frac{1}{2n+2}\,\frac{1}{2n+2}-\frac{1}{2n-1}\,\frac{1}{2n-1}-\frac{1}{2n+2}\,\frac{1}{2n+2}=-\frac{1}{2n-1}\,\frac{1}{2n-1};$$

and when n is even,

$$a = -\frac{1}{2n+1}\,\frac{1}{2n+1}.$$

In the first summation we are adding $4n'+4$ constituents from one end of the column or series yielding the $(4n+1)$-th term of the product; in the second summation we are adding $4n'+4$ constituents from the other end. In the middle of the series the sets of constituents will overlap, if n is odd, and will not touch if n is even. When n is odd, then a represents the sum of the constituents which have been counted twice; when n is even, then a stands for the negative value of the constituent not counted at all. Each of these constituents is of the degree -2 with respect to n and as their number is finite, their sum cannot be of a higher degree with respect to n than the degree -2.

The $(4n+1)$-th term, simplified, is

$$2\sum_{p=0}^{p=n'}\left(\frac{1}{4p+1}\,\frac{-4}{(4n-4p+1)(4n-4p-3)}\right)-a.$$

For any positive integral value of n the fraction $4/(4n-4p+1)(4n-4p-3)$ has its maximum value when $p=n'$. Writing n' for p, we obtain a fraction which is less than $1/n^2$. Hence for any value of p,

$$\frac{1}{4p+1}\,\frac{4}{(4n-4p+1)(4n-4p-3)} < \frac{1}{4p+1}\,\frac{1}{n^2}.$$

It is easy to see that

$$\sum_{n=1}^{2n}\frac{1}{n} < a\log(2n),$$

where a is a constant greater than $3/2\log 2$. Hence

$$\sum_{p=0}^{p=n'}\frac{1}{4p+1} < a\log(2n),$$

and

$$\sum_{p=0}^{p=n'}\frac{1}{4p+1}\,\frac{4}{(4n-4p+1)(4n-4p-3)} < \sum_{p=0}^{p=n'}\frac{1}{4p+1}\,\frac{1}{n^2} < \frac{a\log(2n)}{n^2}.$$

But a number m can be found, such that, when $n>m$, we have

$$\frac{a\log(2n)}{n^2} < \frac{1}{n(\log n)^\theta},$$

where $\theta>1$. Therefore, for $n>m$, the $(4n+1)$-th term is numerically less than the nth term of a series known to be absolutely convergent.

The $(4n+2)$-th term of the product-series is

$$\sum_{p=0}^{p=n'}\Big(-\frac{1}{4n-4p+4}\,\frac{1}{4p+1}+\frac{1}{4n-4p+1}\,\frac{1}{4p+4}+\frac{1}{4n-4p}\,\frac{1}{4p+1}-\frac{1}{4n-4p-3}\,\frac{1}{4p+4}\Big)$$
$$+\sum_{p=0}^{p=n'}\Big(+\frac{1}{4n-4p+4}\,\frac{1}{4p+1}-\frac{1}{4n-4p+1}\,\frac{1}{4p+4}-\frac{1}{4n-4p}\,\frac{1}{4p+1}+\frac{1}{4n-4p-3}\,\frac{1}{4p+4}\Big)-a,$$

where n' is defined as before and where

$$a=\frac{1}{2n+2}\,\frac{1}{2n-1}-\frac{1}{2n-1}\,\frac{1}{2n+2}=0,$$

when n is odd, and

$$a=+\frac{1}{2n+4}\,\frac{1}{2n+1}-\frac{1}{2n+1}\,\frac{1}{2n+4}=0,$$

when n is even. We see that the entire $(4n+2)$-th term is identically equal to zero.

The $(4n+3)$-th term of the product-series is

$$\sum_{p=0}^{p=n'}\Big(+\frac{1}{4n-4p+1}\,\frac{1}{4p+1}-\frac{1}{4n-4p+4}\,\frac{1}{4p+4}-\frac{1}{4n-4p+1}\,\frac{1}{4p+1}+\frac{1}{4n-4p}\,\frac{1}{4p+4}\Big)$$
$$+\sum_{p=0}^{p=n'}\Big(-\frac{1}{4n-4p+1}\,\frac{1}{4p+1}-\frac{1}{4n-4p+4}\,\frac{1}{4p+4}+\frac{1}{4n-4p+1}\,\frac{1}{4p+1}+\frac{1}{4n-4p}\,\frac{1}{4p+4}\Big)-a$$
$$=2\sum_{p=0}^{p=n'}\Big(\frac{1}{4p+4}\,\frac{4}{(4n-4p+4)(4n-4p)}\Big)-a,$$

where

$$a=\frac{1}{2n+2}\,\frac{1}{2n+2},$$

when n is odd, and

$$a=-\frac{1}{2n+1}\,\frac{1}{2n+1}+\frac{1}{2n+4}\,\frac{1}{2n+4}+\frac{1}{2n+1}\,\frac{1}{2n+1},$$

when n is even. The reasoning which we applied to the $(4n+1)$-th term will show that this term, too, when $n>m$, is numerically less than $1/n(\log n)^{\theta}$, $\theta>1$.

We have now shown that the $4n$-th term and the three immediately succeeding terms are either identically zero, or, for $n>m$, numerically less than $1/n(\log n)^{\theta}$. Hence the product of the series S_1 and S_2 converges absolutely.

Again, take the conditionally convergent series, obtained by dropping the parentheses in the series:

$$T_1 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{6p+1}-\frac{1}{6p+2}+\frac{1}{6p+3}-\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}\right)$$

and the divergent series, obtained by dropping the parentheses in the series:

$$T_2 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}+\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}\right).$$

The $6n$-th term of their product, viz.,

$$\sum_{p=0}^{p=n-1}\left(-\frac{1}{6n-6p+3}\frac{1}{6p+1}+\frac{1}{6n-6p+2}\frac{1}{6p+2}+\frac{1}{6n-6p+1}\frac{1}{6p+3}\right.$$
$$\left.+\frac{1}{6n-6p+3}\frac{1}{6p+1}-\frac{1}{6n-6p+2}\frac{1}{6p+2}-\frac{1}{6n-6p+1}\frac{1}{6p+3}\right),$$

is seen to be identically zero.

The $(6n+1)$-th term of the product of T_1 and T_2 is

$$\sum_{p=0}^{p=n'}\left(+\frac{1}{6n-6p+1}\frac{1}{6p+1}-\frac{1}{6n-6p-3}\frac{1}{6p+2}-\frac{1}{6n-6p-4}\frac{1}{6p+3}\right.$$
$$\left.-\frac{1}{6n-6p-5}\frac{1}{6p+1}+\frac{1}{6n-6p-3}\frac{1}{6p+2}+\frac{1}{6n-6p-4}\frac{1}{6p+3}\right)$$
$$+\sum_{p=0}^{p=n'}\left(+\frac{1}{6n-6p+1}\frac{1}{6p+1}+\frac{1}{6n-6p-3}\frac{1}{6p+2}+\frac{1}{6n-6p-4}\frac{1}{6p+3}\right.$$
$$\left.-\frac{1}{6n-6p-5}\frac{1}{6p+1}-\frac{1}{6n-6p-3}\frac{1}{6p+2}-\frac{1}{6n-6p-4}\frac{1}{6p+3}\right)-a$$
$$=2\sum_{p=0}^{p=n'}\left(\frac{1}{6p+1}\frac{-6}{(6n-6p+1)(6n-6p-5)}\right)-a,$$

where n' is defined as before and where

$$a = -\frac{1}{3n}\frac{1}{3n-1} - \frac{1}{3n-1}\frac{1}{3n} - \frac{1}{3n-2}\frac{1}{3n-2} + \frac{1}{3n}\frac{1}{3n-1} + \frac{1}{3n-1}\frac{1}{3n}$$
$$= -\frac{1}{3n-2}\frac{1}{3n-2},$$

when n is odd, and $a = -1/(3n+1)(3n+1)$, when n is even. By repeating the reasoning used in previous cases we find that this term, for $n > m$, is numerically less than $1/n(\log n)^\theta$, $\theta > 1$.

The $(6n+3)$-th, $(6n+4)$-th, and $(6n+5)$-th terms in the product of T_1 and T_2, when subjected to this course of reasoning, all yield the same conclusion. The $(6n+2)$-th term is identically zero. Hence the product-series is absolutely convergent.

§ 3. *The general method.*

The method by which the above factor-series were constructed is simple. The harmonic series was modified by adopting some simple law of succession of signs. To render the series conditionally convergent, make the number of + terms in a set equal to the number of − terms. If these numbers are unequal, then the series is divergent. Then add suitable finite numbers to the denominators of some of the terms in the set, so that, when the product of the two series is formed, the constituents of any set of the nth term in the product, when added, yield a fractional sum having denominators whose degree in n is higher by two than the degree of the numerators. We were able to prove the absolute convergence from the fact that in each complete set of constitutents of the nth term in the product-series there are as many positive constituents as there are negative ones. To secure this even distribution of signs among the constituents for the sets in every term of the product, it is necessary that in one of the factor-series the number of + terms in any set be equal to the number of − terms in that set. This condition is not sufficient, because among the terms in the product some might have a preponderance of + constituents and others a preponderance of − constituents. This happens, for instance, when we multiply the series + + − − + + − − etc. by itself.

In the proper selection of signs the following rule, which we give without proof, may be of service. Let a, denoting the number of terms in each set of the series $\sum u_m$, be an even number and equal to $2c$; also let the first c terms in each set be +, the remaining terms −. Let the number of terms in each set of the second series, $\sum v_m$, be b. Then, in order to secure as many + constituents in each set of a term of the product of the two series $\sum u_m$ and $\sum v_m$, as there are negative constituents, it is sufficient that d should be a factor of c, where d is the smallest value, numerically, satisfying the congruence $b \equiv d \pmod{a}$.

In explaining the further process of constructing our series we shall start out with a particular selection of signs, but the process is the same as for any other selection, similarly made. Let $a=4$, $b=3$, then $c=2$ and $d=-1$, a factor of c. Accordingly, the signs of terms in each set of the first series are $++--$. Let the signs of the b terms in each set of $\sum v_m$ be $++-$. We have now the two factor-series obtained by dropping the parentheses from the series on the right hand sides of the formulas:

$$\sum u_m \equiv \sum_{p=1}^{p=\infty}\left(\frac{1}{4p+a_1}+\frac{1}{4p+1+a_2}-\frac{1}{4p+2+a_3}-\frac{1}{4p+3+a_4}\right)$$

and

$$\sum v_m \equiv \sum_{p=1}^{p=\infty}\left(\frac{1}{3p+\beta_1}+\frac{1}{3p+1+\beta_2}-\frac{1}{3p+2+\beta_3}\right),$$

in which the Greek letters are quantities to be chosen, if possible, so that the product of the two series shall be absolutely convergent.

For brevity, we shall, in the series $\sum u_m$, represent by $\{u_{ap}+a\}$ a fraction $1/(ap+a)$, and, similarly, in the series $\sum v_m$, by $\{v_{bp}+\beta\}$ a fraction $1/(bp+\beta)$. Accordingly we have, dropping the parentheses as before,

$$\sum u_m \equiv \sum_{p=1}^{p=\infty}(\{u_{4p}+a_1\}+\{u_{4p}+1+a_2\}-\{u_{4p}+2+a_3\}-\{u_{4p}+3+a_4\})$$

and

$$\sum v_m \equiv \sum_{p=1}^{p=\infty}(\{v_{3p}+\beta_1\}+\{v_{3p}+1+\beta_2\}-\{v_{3p}+2+\beta_3\}).$$

In the product of $\sum u_m$ and $\sum v_m$, a set will consist of 12 constituents. The *first* set in the $(12n)$-th term will be (since $\{v_{12}-9+\beta_1\}\equiv\{v_3+\beta_1\}$, and similarly in other cases)

$$\text{(I)}\quad\begin{vmatrix}
-\{u_{12n}+3+a_4''\}\{v_{12}-9+\beta_1\}-\{u_{12n}+2+a_3''\}\{v_{12}-8+\beta_2\}\\
-\{u_{12n}+1+a_2''\}\{v_{12}-7+\beta_3\}+\{u_{12n}+a_1''\}\{v_{12}-6+\beta_1'\}\\
-\{u_{12n}-1+a_4'\}\{v_{12}-5+\beta_2'\}+\{u_{12n}-2+a_3'\}\{v_{12}-4+\beta_3'\}\\
+\{u_{12n}-3+a_2'\}\{v_{12}-3+\beta_1''\}+\{u_{12n}-4+a_1'\}\{v_{12}-2+\beta_2''\}\\
+\{u_{12n}-5+a_4\}\{v_{12}-1+\beta_3''\}-\{u_{12n}-6+a_3\}\{v_{12}+\beta_1'''\}\\
+\{u_{12n}-7+a_2\}\{v_{12}+1+\beta_2'''\}-\{u_{12n}-8+a_1\}\{v_{12}+2+\beta_3'''\}.
\end{vmatrix}$$

Here each of the quantities β_1, β_2, β_3 occurs four times. We have broadened our assumptions by attaching a single accent to each letter the first time it is repeated and a double and triple accent the second and third time it is repeated. Accordingly β_1, β_1', β_1'', β_1''', β_2, β_2', β_2'', β_2''', β_3, β_3', β_3'', β_3''' may in general represent each a different number. Similarly, the letters a_1, a_2, a_3, a_4 receive a single accent the first time they are repeated in a set of constituents and a double accent the second time they are repeated.

We shall limit ourselves to the case where, in any term of the product series, each set, when multiplied by n, yields a product that converges toward zero when $n=\infty$. Accordingly, if the set I be multiplied by n, we get, for $n=\infty$ (supposing the Greek letters to be positive constants, or variables which approach positive finite limits), the first of the two following equations:

$$\text{(II)}\quad \begin{cases} -\{v_{12}-9+\beta_1\}-\{v_{12}-8+\beta_2\}-\{v_{12}-7+\beta_3\}+\{v_{12}-6+\beta_1'\} \\ -\{v_{12}-5+\beta_2'\}+\{v_{12}-4+\beta_3'\}+\{v_{12}-3+\beta_1''\}+\{v_{12}-2+\beta_2''\} \\ +\{v_{12}-1+\beta_3''\}-\{v_{12}+\beta_1'''\}+\{v_{12}+1+\beta_2'''\}-\{v_{12}+2+\beta_3'''\}=0, \\ +\{v_{12}-9+\beta_1\}-\{v_{12}-8+\beta_2\}+\{v_{12}-7+\beta_3\}+\{v_{12}-6+\beta_1'\} \\ +\{v_{12}-5+\beta_2'\}+\{v_{12}-4+\beta_3'\}-\{v_{12}-3+\beta_1''\}+\{v_{12}-2+\beta_2''\} \\ -\{v_{12}-1+\beta_3''\}-\{v_{12}+\beta_1'''\}-\{v_{12}+1+\beta_2'''\}-\{v_{12}+2+\beta_3'''\}=0. \end{cases}$$

The second equation in II is got from the $(12n+1)$-th term by the same procedure. All the even terms between the $(12n)$-th and the $(12n+12)$-th yield the same equation as the $(12n)$-th term; all the odd terms yield the same equation as the $(12n+1)$-th term.

The *last* set in the $(12n)$-th term of the product is

$$\text{(III)}\quad \begin{cases} -\{v_{12n}+2+\beta_3'''\}\{u_{12}-8+a_1\}+\{v_{12n}+1+\beta_2'''\}\{u_{12}-7+a_2\} \\ -\{v_{12n}-1+\beta_1'''\}\{u_{12}-6+a_3\}+\{v_{12n}-1+\beta_3''\}\{u_{12}-5+a_4\} \\ +\{v_{12n}-2+\beta_2''\}\{u_{12}-4+a_1'\}+\{v_{12n}-3+\beta_1''\}\{u_{12}-3+a_2'\} \\ +\{v_{12n}-4+\beta_3'\}\{u_{12}-2+a_3'\}-\{v_{12n}-5+\beta_2'\}\{u_{12}-1+a_4'\} \\ +\{v_{12n}-6+\beta_1'\}\{u_{12}-a_1''\}-\{v_{12n}-7+\beta_3\}\{u_{12}+1+a_2''\} \\ -\{v_{12n}-8+\beta_2\}\{u_{12}+2+a_3''\}-\{v_{12n}-9+\beta_1\}\{u_{12}+3+a_4''\}. \end{cases}$$

Assuming set III, when multiplied by n, to yield a product which converges toward zero, when $n=\infty$, we get the first of the following three equations:

$$\text{(IV)}\quad \begin{cases} -\{u_{12}-8+a_1\}+\{u_{12}-7+a_2\}-\{u_{12}-6+a_3\}+\{u_{12}-5+a_4\} \\ +\{u_{12}-4+a_1'\}+\{u_{12}-3+a_2'\}+\{u_{12}-2+a_3'\}-\{u_{12}-1+a_4'\} \\ +\{u_{12}+a_1''\}-\{u_{12}+1+a_2''\}-\{u_{12}+2+a_3''\}-\{u_{12}+3+a_4''\}=0, \\ +\{u_{12}-8+a_1\}-\{u_{12}-7+a_2\}-\{u_{12}-6+a_3\}-\{u_{12}-5+a_4\} \\ -\{u_{12}-4+a_1'\}+\{u_{12}-3+a_2'\}-\{u_{12}-2+a_3'\}+\{u_{12}-1+a_4'\} \\ +\{u_{12}+a_1''\}+\{u_{12}+1+a_2''\}+\{u_{12}+2+a_3''\}-\{u_{12}+3+a_4''\}=0, \\ +\{u_{12}-8+a_1\}+\{u_{12}-7+a_2\}+\{u_{12}-6+a_3\}-\{u_{12}-5+a_4\} \\ +\{u_{12}-4+a_1'\}-\{u_{12}-3+a_2'\}-\{u_{12}-2+a_3'\}-\{u_{12}-1+a_4'\} \\ -\{u_{12}+a_1''\}+\{u_{12}+1+a_2''\}-\{u_{12}+2+a_3''\}+\{u_{12}+3+a_4''\}=0. \end{cases}$$

The second and third equations in IV are derived from the last set in the $(12n+1)$-th and $(12n+2)$-th term, respectively. The $(12n+3)$-th,

$(12n+6)$-th, and $(12n+9)$-th terms yield the same equation in IV as the $(12n)$-th term; the $(12n+4)$-th, $(12n+7)$-th, $(12n+10)$-th terms the same as the $(12n+1)$-th term; the $(12n+5)$-th, $(12n+8)$-th, $(12n+11)$-th the same as the $(12n+2)$-th term.

The equations II, when simplified and by the substitution of v_{12p} for v_{12} made to apply to other sets in the terms of the product, are

$$\text{(V)}\quad \begin{cases} -\{v_{12p}-8+\beta_2\}+\{v_{12p}-6+\beta_1'\}+\{v_{12p}-4+\beta_3'\} \\ \qquad +\{v_{12p}-2+\beta_2''\}-\{v_{12p}+\beta_1'''\}-\{v_{12p}+2+\beta_3'''\}=0, \\ \{v_{12p}-9+\beta_1\}+\{v_{12p}-7+\beta_3\}+\{v_{12p}-5+\beta_2'\} \\ \qquad -\{v_{12p}-3+\beta_1''\}-\{v_{12p}-1+\beta_3''\}-\{v_{12p}+1+\beta_2'''\}=0. \end{cases}$$

The equations IV, when simplified, u_{12p} being written for u_{12}, become

$$\text{(VI)}\quad \begin{cases} \{u_{12p}-7+a_2\}+\{u_{12p}-4+a_1'\}-\{u_{12p}-1+a_4'\}-\{u_{12p}+2+a_3''\}=0, \\ \{u_{12p}-8+a_1\}-\{u_{12p}-5+a_4\}-\{u_{12p}-2+a_3'\}+\{u_{12p}+1+a_2''\}=0, \\ \{u_{12p}-6+a_3\}-\{u_{12p}-3+a_2'\}-\{u_{12p}+a_1''\}+\{u_{12p}+3+a_4''\}=0. \end{cases}$$

If we aim to assign to the Greek letters constant values, then, in every equation in V each fraction must admit of being associated singly with another fraction in that equation which is numerically equal to it, but of opposite sign. This relation is brought about by the following six equations:

$$\text{(VII)}\quad \begin{cases} \{v_{12p}-9+\beta_1\}=\{v_{12p}-3+\beta_1''\}, & \{v_{12p}-6+\beta_1'\}=\{v_{12p}+\beta_1'''\}, \\ \{v_{12p}-8+\beta_2\}=\{v_{12p}-2+\beta_2''\}, & \{v_{12p}-5+\beta_2'\}=\{v_{12p}+1+\beta_2''\}, \\ \{v_{12p}-7+\beta_3\}=\{v_{12p}-1+\beta_3''\}, & \{v_{12p}-4+\beta_3'\}=\{v_{12p}+2+\beta_3'''\}. \end{cases}$$

The same considerations applied to equations VI yield the following six relations:

$$\text{(VIII)}\quad \begin{cases} \{u_{12p}-8+a_1\}=\{u_{12p}-2+a_3'\}, & \{u_{12p}-5+a_4\}=\{u_{12p}+1+a_2''\}, \\ \{u_{12p}-7+a_2\}=\{u_{12p}-1+a_4'\}, & \{u_{12p}-4+a_1'\}=\{u_{12p}+2+a_3''\}, \\ \{u_{12p}-6+a_3\}=\{u_{12p}+a_1''\}, & \{u_{12p}-3+a_2'\}=\{u_{12p}+3+a_4''\}. \end{cases}$$

Applying relations VIII, one may write the series $\sum u_m$ thus, the parentheses being suppressed:

$$\text{(IX)}\quad \begin{cases} \sum u_m \equiv \sum_{p=1}^{p=\infty}(\{u_{12p}-8+a_1\}+\{u_{12p}-7+a_2\}-\{u_{12p}-6+a_3\} \\ \qquad -\{u_{12p}-5+a_4\}+\{u_{12p}-4+a_1'\}+\{u_{12p}-3+a_2'\} \\ \qquad -\{u_{12p}-8+a_1\}-\{u_{12p}-7+a_2\}+\{u_{12p}-6+a_3\} \\ \qquad +\{u_{12p}-5+a_4\}-\{u_{12p}-4+a_1'\}-\{u_{12p}-3+a_2'\}). \end{cases}$$

Applying relations VII, one may write the series $\sum v_m$ thus, the parentheses being suppressed:

$$(\mathrm{X}) \quad \left\{ \begin{aligned} \sum v_m \equiv \sum_{p=1}^{p=\infty} (& \{v_{12p} - 9 + \beta_1\} + \{v_{12p} - 8 + \beta_2\} - \{v_{12p} - 7 + \beta_3\} \\ & + \{v_{12p} - 6 + \beta_1'\} + \{v_{12p} - 5 + \beta_2'\} - \{v_{12p} - 4 + \beta_3'\} \\ & + \{v_{12p} - 9 + \beta_1\} + \{v_{12p} - 8 + \beta_2\} - \{v_{12p} - 7 + \beta_3\} \\ & + \{v_{12p} - 6 + \beta_1'\} + \{v_{12p} - 5 + \beta_2'\} - \{v_{12p} - 4 + \beta_3'\}). \end{aligned} \right.$$

Series IX is conditionally convergent; series X divergent. In both series any constant positive values may be assigned to the Greek letters. The product of the two series is absolutely convergent. This may be shown by examining the $12n$-th term and the eleven immediately succeeding terms of the product-series in the same way as we did the terms in the product of S_1 and S_2. We shall here write down the expression for the $(12n+1)$-th term only, viz.,

$$\begin{aligned} & \sum_{p=1}^{p=n'} (+ \{u_{12n-12p} + 16 + \alpha_1\}\{v_{12p} - 9 + \beta_1\} - \{u_{12n-12p} + 9 + \alpha_2'\}\{v_{12p} - 8 + \beta_2\} \\ & \quad + \{u_{12n-12p} + 8 + \alpha_1'\}\{v_{12p} - 7 + \beta_3\} + \{u_{12n-12p} + 7 + \alpha_4\}\{v_{12p} - 6 + \beta_1'\} \\ & \quad + \{u_{12n-12p} + 6 + \alpha_3\}\{v_{12p} - 5 + \beta_2'\} + \{u_{12n-12p} + 5 + \alpha_2\}\{v_{12p} - 4 + \beta_3'\} \\ & \quad - \{u_{12n-12p} + 4 + \alpha_1\}\{v_{12p} - 9 + \beta_1\} + \{u_{12n-12p} + 9 + \alpha_2'\}\{v_{12p} - 8 + \beta_2\} \\ & \quad - \{u_{12n-12p} + 8 + \alpha_1'\}\{v_{12p} - 7 + \beta_3\} - \{u_{12n-12p} + 7 + \alpha_4\}\{v_{12p} - 6 + \beta_1'\} \\ & \quad - \{u_{12n-12p} + 6 + \alpha_3\}\{v_{12p} - 5 + \beta_2'\} - \{u_{12n-12p} + 5 + \alpha_2\}\{v_{12p} - 4 + \beta_3'\}) \\ & + \sum_{p=1}^{p=n'} (+ \{v_{12n-12p} + 15 + \beta_1\}\{u_{12p} - 8 + \alpha_1\} - \{v_{12n-12p} + 8 + \beta_3'\}\{u_{12p} - 7 + \alpha_2\} \\ & \quad - \{v_{12n-12p} + 7 + \beta_2'\}\{u_{12p} - 6 + \alpha_3\} - \{v_{12n-12p} + 6 + \beta_1'\}\{u_{12p} - 5 + \alpha_4\} \\ & \quad - \{v_{12n-12p} + 5 + \beta_3\}\{u_{12p} - 4 + \alpha_1'\} + \{v_{12n-12p} + 4 + \beta_2\}\{u_{12p} - 3 + \alpha_2'\} \\ & \quad - \{v_{12n-12p} + 3 + \beta_1\}\{u_{12p} - 8 + \alpha_1\} + \{v_{12n-12p} + 8 + \beta_3'\}\{u_{12p} - 7 + \alpha_2\} \\ & \quad + \{v_{12n-12p} + 7 + \beta_2'\}\{u_{12p} - 6 + \alpha_3\} + \{v_{12n-12p} + 6 + \beta_1'\}\{u_{12p} - 5 + \alpha_4\} \\ & \quad + \{v_{12n-12p} + 5 + \beta_3\}\{u_{12p} - 4 + \alpha_1'\} - \{v_{12n-12p} + 4 + \beta_2\}\{u_{12p} - 3 + \alpha_2'\}) \\ & - \alpha \\ & \quad = -12 \sum_{p=1}^{p=n'} \{v_{12p} - 9 + \beta_1\}\{u_{12n-12p} + 16 + \alpha_1\}\{u_{12n-12p} + 4 + \alpha_1\} \\ & \qquad - 12 \sum_{p=1}^{p=n'} \{u_{12p} - 8 + \alpha_1\}\{v_{12n-12p} + 15 + \beta_1\}\{v_{12n-12p} + 3 + \beta_1\} - \alpha, \end{aligned}$$

where $n' = \frac{1}{2}n$ when n is even, and $n' = \frac{1}{2}(n+1)$ when n is odd, and where α is used in the same way as before. In this case α consists of one constituent when n is even, and of eleven constituents when n is odd. By the very same reasoning as that applied in the series S_1 and S_2 it follows that the $(12n+1)$-th

term of the product of IX and X, for $n > m$, has a numerical value less than $1/n(\log n)^{\theta}$, where $\theta > 1$. The two factor-series IX and X differ from the pairs S_1, S_2 and T_1, T_2 in this—that (suppressing in each series each parenthesis which encloses a set) corresponding terms in IX and X need not have the same numerical values. However, the three pairs of factor-series have in common one property, to which allusion has not yet been made; viz., in one of the series of each pair, the sum of the terms in each set is identically zero, so that the sum of the series is itself zero.

COLORADO COLLEGE, COLORADO SPRINGS.

SETS OF COINCIDENCE* POINTS ON THE NON-SINGULAR CUBICS OF A SYZYGETIC SHEAF†

BY

M. B. PORTER

I.

If $C_3 = 0$ be any non-singular cubic and

$$u = \int_{ab}^{xy} du$$

be an everywhere finite integral attached to this curve, the point (a, b) being a point of inflexion, the necessary and sufficient condition that $3m$ points $P_1 \cdots P_{3m}$ of C_3 lie also on an m-ic is given by ABEL's theorem, namely:

$$\sum_1^{3m} u_i \equiv 0 \pmod{\omega, \omega'}, \tag{1}$$

where ω and ω' are the moduli of periodicity of u and

$$u_i = \int_{ab}^{x_i y_i} du \qquad (i = 1, \cdots, 3m),$$

the points P_i being the points (x_i, y_i).

If $3m - s - 1$ of these points P_i be taken as fixed and we suppose the m-ic to have a contact of order $s - 1$ at u_1, the m-ic will still cut C_3 at a point u_2 dependent on u_1 and by (1) we have:

$$su_1 + u_2 \equiv c \pmod{\omega, \omega'},$$

where c denotes the negative of the sum of the u's of the arbitrarily chosen *fixed* points.

We consider the following sequence of n such congruences:

$$\left.\begin{array}{ll} su_1 + u_2 & \equiv c \\ su_2 + u_3 & \equiv c \\ \vdots \quad\quad \vdots & \quad\quad \vdots \\ su_n + u_{n+1} & \equiv c \end{array}\right\} \pmod{\omega, \omega'};$$

* Points where a cubic can have an eighth order contact with cubics of the syzygetic sheaf are called by HALPHEN coincidence points of the cubic.

† Presented to the Society at the New York meeting, June 29, 1900, under a different title. Received for publication November 16, 1900.

if we require that $u_{n+1} \equiv u_1$, i. e., that the curvilinear polygon of points of parameters $u_1 \cdots u_{n+1}$ be a closed polygon,* the necessary and sufficient condition will be:

$$[s^n - (-1)^n]u_1 \equiv c' \pmod{\omega, \omega'}. \tag{2}$$

In case $m = s = 1$, we have a rectilinear polygon of n sides inscribed and circumscribed in C_3, (2) becoming in this case,

$$[2^n - (-1)^n]\, u_1 \equiv 0 \pmod{\omega, \omega'}. \tag{2'}$$

The number of actual n-gons is given by a formula of PICQUET* and it is evident from the nine perspective transformations of C_3 into itself that any one of these n-gons will be perspective with some other n-gon relative to an inflexion point.

HALPHEN‡ has investigated the points $u = (n'\omega + n''\omega')/3m$ which contain as special cases the vertices of these rectilinear n-gons for the non-singular cubics of the syzygetic sheaf $x^3 + y^3 + z^3 + 6axyz = 0$, and finds that the locus of these points as a varies, which is evidently a combinant of the sheaf, consists of eight irreducible curves when m is divisible by 3, and of nine such curves when m is not so divisible.

II.

It is the purpose of this section to consider certain geometrical relations that subsist between an inflexion triangle and its associated group of in- and circumscribed rectilinear triangles. By (2') the number of such triangles is 24.

---

*The *Schliessungsproblem* here suggested is a slight generalization of that considered by PICQUET (Journal de l'École Polytechnique, vol. 54, 1884). The method of enumeration used by PICQUET is of course applicable here, but the number-theoretic results arrived at are completely general. Thus for n prime the number of n-gons on the part of the curve corresponding to ω will be:

$$\frac{s^n - s}{n}, \tag{A}$$

since $s+1$ values of u must be rejected as corresponding to the s-contact points. In case n be not prime, to obtain the number of n-gons it would be necessary to reject besides the $s+1$ s-contact points all solutions corresponding to polygons whose number of sides divides n. Denoting by $\phi(s, m)$ the number of vertices of such m-gons on the part of the curve corresponding to ω we have:

$$s^n - (-1)^n - s - 1 = \sum_{m'} \phi(s, m') \qquad (1 < m' \leqq n), \tag{B}$$

where the summation is for the different factors m' of n.

By [B] for $n+2$ we get:

$$\phi(s, n) = s^n - \sum s^{\frac{n}{a_i}} + \sum s^{\frac{n}{a_i a_j}} - \cdots \pm s^{\frac{n}{a_1 a_2 \cdots a_\nu}} = F(s, n),$$

$a_1 a_2 \cdots a_\nu$ denoting the different prime factors of n. For $n = 2$ we have $\phi = F - 2$.

The expressions [A] and [B] may be regarded as establishing FERMAT'S theorem and its generalization.

‡ *Recherches sur les courbes planes du troisième degré*, Mathematische Annalen, vol. 15.

These twenty-four triangles can be divided into four groups of six each, each group being associated with one of the four inflexion triangles. Further the six triangles of any group can be paired in nine ways so that each pair shall be perspective in three ways, the centers of perspectivity for each pair being the three inflexion points lying on a side of the associated inflexion triangle.

Thus for the inflexion triangle whose sides may be denoted by

$$\left(\omega + \frac{n\omega'}{3}, \quad \frac{\omega + n\omega'}{3}, \quad \frac{2\omega + n\omega'}{3}\right) \qquad (n = 0, 1, 2),$$

the scheme is as follows:

ω	$\frac{\omega}{3}$	$\frac{2\omega}{3}$
[A] $\frac{\omega}{9} + \frac{n\omega'}{3}$	[B] $-\frac{2\omega}{9} + \frac{n\omega'}{3}$	[C] $\frac{4\omega}{9} + \frac{n\omega'}{3}$
[A′] $-\frac{\omega}{9} - \frac{n\omega'}{3}$	[B′] $\frac{2\omega}{9} - \frac{n\omega'}{3}$	[C′] $-\frac{4\omega}{9} - \frac{n\omega'}{3}$

$\omega + \frac{\omega'}{3}$	$\frac{\omega + \omega'}{3}$	$\frac{2\omega + \omega'}{3}$
$\frac{\omega}{9} + \frac{n\omega'}{3}$	$-\frac{2\omega}{9} + \frac{n\omega'}{3}$	$\frac{4\omega}{9} + \frac{n\omega}{3}$
$-\frac{\omega}{9} - \frac{(n+1)\omega'}{3}$	$\frac{2\omega}{9} - \frac{(n+1)\omega'}{3}$	$-\frac{4\omega}{9} - \frac{(n+1)\omega'}{3}$

$\omega + \frac{2\omega'}{3}$	$\frac{\omega + 2\omega'}{3}$	$\frac{2\omega + 2\omega'}{3}$
$\frac{\omega}{9} + \frac{n}{3}\omega'$	$-\frac{2\omega}{9} + \frac{n\omega'}{3}$	$\frac{4\omega}{9} + \frac{n\omega'}{3}$
$-\frac{\omega}{9} - \frac{(n-1)\omega'}{3}$	$\frac{2\omega}{9} - \frac{(n-1)\omega'}{3}$	$-\frac{4\omega}{9} - \frac{(n-1)\omega'}{3}$

where ABC is perspective with $A'B'C'$, with $B'C'A'$, and with $C'A'B'$, the centers of perspectivity being ω, $2\omega/3$, and $\omega/3$ respectively. Thus the nine intersections of the sides of these two triangles lie three by three on the harmonic polar of ω, $2\omega/3$, $\omega/3$.

Referred to one of these in-circumscribed triangles, as $\omega/9$, $-2\omega/9$, $4\omega/9$, the equation of C_3 can, by a suitable choice of the constants of projection, be written:

[A]* $$x^2z + y^2x + z^2y + 2\eta xyz = 0,$$

the other sheaf of cubics in-circumscribed in this same triangle being

[A′] $$y^2z + z^2x + x^2y + 2\eta xyz = 0,$$

* This equation will be referred to indifferently as sheaf [A] or cubic (of sheaf) [A].

where [A′] is derived from [A] by the involutory transformation

$$x : y : z = y'z' : x'z' : x'y' .$$

The two cubics [A] and [A′] are evidently transformed into themselves by the three transformations:

$$[B] \qquad x : y : z = y' : z' : x' , \quad z' : x' : y' , \quad x' : y' : z' ;$$

the remainder of the 18 linear transformations of C_3 into itself depend on the parameter η. If a straight line be drawn through any vertex of the triangle of reference and the two variable points of intersection with C_3 be put in correspondence we have the three involutory transformations of [A] or [A′] into themselves given by

$$[C] \qquad x : y : z = y'z' : x'y' : x'z' , \quad x'y' : x'z' : y'z' , \quad x'z' : y'z' : x'y' .$$

Under transformation [C] the nine inflexion points will be interchanged with the vertices of three in-circumscribed triangles and all the remaining vertices of in-circumscribed triangles will be merely interchanged. Thus, of the covariants whose vanishing gives the vertices of the triangles in question, three will go over into the sides of the associated inflexion triangle and must therefore be conics circumscribing the triangle of reference, while the remaining covariants will be merely interchanged or transformed into themselves under the transformation [C].

To obtain these conics it will thus only be necessary to determine the equation of the sides of the associated inflexion triangle.

The Hessian of [A] is

$$x^3 + y^3 + z^3 - 3xyz - \eta^3(x^2z + y^2x + z^2y + 2\eta xyz) = 0 ;$$

thus all the cubics of either sheaf [A] or [A′] have the fixed inflexion triangle:

$$x^3 + y^3 + z^3 - 3xyz = \begin{vmatrix} y & z & x \\ x & y & z \\ z & x & y \end{vmatrix} = (x + y + z)(x + \epsilon y + \epsilon^2 z)(x + \epsilon^2 y + \epsilon z) ,$$

where $\epsilon^3 = 1$. By the application to this inflexion triangle of the transformation [C] we find that the vertices of three of the in-circumscribed triangles of the sheaves [A], [A′] lie on the conics:

$$(\alpha) \qquad xy + yz + xz = 0 ,$$

$$(\beta) \qquad xy + \epsilon yz + \epsilon^2 xz = 0 ,$$

$$(\gamma) \qquad xy + \epsilon^2 yz + \epsilon xz = 0 .$$

Each of these conics has a double contact with two sides of the associated inflexion triangle, the third side being the chord of contact. The Hessian of the net determined by (α), (β), (γ) being the sides of the triangle of reference.*

* The conics α, β, γ may be obtained directly without recourse to the Hessian; for if $ax + by + cz = 0$ be the equation of a side of the associated inflexion triangle we must have, applying [C], $axz + byz + cxy \equiv \rho(ayz + bxy + cxz)$, so that $\rho = 1^{1/3}$, i. e., $a = b = c$ or $a = \epsilon b = \epsilon^2 c$ or $a = \epsilon^2 b = \epsilon c$.

The vertex of any in-circumscribed triangle is the tangential of three points whose affixes are given by $\bar{u} = (n\omega + n'\omega')/18$. The loci of the nine points associated with the vertices of the triangle of reference on which the $\bar{u}$ points lie three by three are:

(α') $\qquad x^2 - yz = 0\,,$

(β') $\qquad y^2 - xz = 0\,,$

(γ') $\qquad z^2 - xy = 0\,.$

These conics are the first polars of the vertices of the fundamental triangle relative to the inflexion triangle, and have precisely the same relation to the fundamental triangle that (α), (β), (γ) have to the inflexion triangle. The Hessian of their net is thus the sides of the inflexion triangle. If the inflexion triangle be taken as the triangle of coördinates, [A] can be written:

$$\bar{x}^3 + \bar{y}^3 + \bar{z}^3 + 6b\bar{x}\bar{y}\bar{z} = 0\,,$$

where

$$\bar{x} = \sqrt[3]{3 + 2\eta}\,(x + y + z)\,,$$

$$\bar{y} = \sqrt[3]{3\epsilon^2 + 2\eta}\,(x + \epsilon^2 y + \epsilon z)\,,$$

$$\bar{z} = \sqrt[3]{3\epsilon + 2\eta}\,(x + \epsilon y + \epsilon^2 z)\,,$$

$$b = -\frac{\eta}{\sqrt{27 + 8\eta^3}}\,.$$

Writing

$$\lambda_1 = \sqrt[3]{3 + 2\eta}\,, \quad \lambda_2 = \sqrt[3]{3\epsilon^2 + 2\eta}\,, \quad \lambda_3 = \sqrt[3]{3\epsilon + 2\eta}\,,$$

we have, as the equation of a side of one of the variable inflexion triangles of [A],

$$(\lambda_1 + \lambda_2 + \lambda_3)x + (\lambda_1 + \epsilon^2\lambda_2 + \epsilon\lambda_3)y + (\lambda_1 + \epsilon\lambda_2 + \epsilon^2\lambda_3)z = 0\,,$$

which envelopes the locus:

$$\rho u = \lambda_1 + \lambda_2 + \lambda_3\,,$$

$$\rho v = \lambda_1 + \epsilon^2\lambda_2 + \epsilon\lambda_3\,,$$

$$\rho w = \lambda_1 + \epsilon\lambda_2 + \epsilon^2\lambda_3\,,$$

or

$$wv^2 + vu^2 + uw^2 = 0\,.$$

This equianharmonic curve of class three is inscribed in all three of the variable inflexion triangles of [A].

The locus of the variable vertices of these inflexion triangles is found to be three sextics.

The equations of the nine joins of the vertices of the triangle of reference and those of the associated inflexion triangle are:

$$x - y = 0\,, \qquad x - \epsilon y = 0\,, \qquad x - \epsilon^2 y = 0\,,$$

$$y - z = 0\,, \qquad y - \epsilon z = 0\,, \qquad y - \epsilon^2 z = 0\,,$$

$$z - x = 0\,, \qquad z - \epsilon x = 0\,, \qquad z - \epsilon^2 x = 0\,.$$

Since any three of these equations, picked out after the manner of a term of a three-row determinant, are simultaneous we have the theorem:

Each in-circumscribed triangle is in six ways perspective with its associated inflexion triangle. *

There remain two more triangles associated with the inflexion triangle under consideration. The loci of these vertices are easily determined by the method employed by HALPHEN. Let $x'y'z'$ be the current coördinates; the tangent at xyz is

$$x'(2xz + y^2 + 2\eta yz) + y'(2xy + z^2 + 2\eta xz) + z'(2yz + x^2 + 2\eta xy) = 0 .$$

If we make the substitution $x' : y' : z' = z : x : y$ which transforms [A] into itself, it is clear that the locus:

$$z(2xz + y^2 + 2\eta yz) + x(2xy + z^2 + 2\eta xz) + y(2yz + x^2 + 2\eta xy) = 0 ,$$

will pass through the vertices of an in-circumscribed triangle. Eliminating η by means of [A] we get:

$$y^4z^2 + z^4x^2 + x^4y^2 - z^3xy^2 - x^2y^3z - x^3yz^2 = 0 ,$$

that is,

$$(y^2z + \epsilon z^2x + \epsilon^2x^2y)(y^2z + \epsilon^2z^2x + \epsilon x^2y) = 0 .\dagger$$

These two equianharmonic cubics are in-circumscribed in three of the inflexion triangles of the sheaf:

$$[D] \qquad x^3 + y^3 + z^3 + 6bxyz = 0 ,$$

and have the fourth inflexion triangle for an inflexion triangle.

It is to be noted that the conics α, β, γ, α', β', γ' are taken together a combinant of the sheaf of cubics [D], there being in all thirty-six such conics, each of which circumscribes one inflexion triangle and has a double contact with two sides of another, the third side of the second triangle being the chord of contact. The rôle played by these conics in the geometry of the sheaf [D] has not, so far as the writer is aware, been investigated.

YALE UNIVERSITY, NEW HAVEN, CONN.

* The same thing is well known to be true for any two inflexion triangles, the six centers of perspectivity here being the vertices of the other two inflexion triangles. The theorem in question may be deduced from the fact that the vertices of the inflexion triangles lie three by three on the harmonic polars.

† These cubics are together with $y^2z + z^2x + x^2y = 0$ a combinant of the sheaf of curves [D] below and determine on it the vertices of nine in-circumscribed triangles. These cubics belong to the set of eight obtained by HALPHEN, loc. cit., and were given by Professor H. S. WHITE, Transactions of the American Mathematical Society, April, 1900.

NOTE ON NON-QUATERNION NUMBER SYSTEMS*

BY

WENDELL M. STRONG

SCHEFFERS† has divided all number systems into the quaternion and non-quaternion systems and has shown that the n fundamental units of a non-quaternion system may be so chosen that the multiplication table takes a particularly simple form, which is in turn characteristic of the non-quaternion systems. In this paper I shall show that the choice of the units may be so regulated that the multiplication table becomes still simpler.

SCHEFFER'S form, which we shall call the *regular form*, has the following characteristic properties:

1°. The units are divided into two essentially different classes, the e's and the η's, with the notation:

$$e_1, e_2, \cdots, e_r; \quad \eta_1, \eta_2, \cdots, \eta_s \qquad (r+s=n).$$

2°. The sum of the η's is the modulus (or idemfactor).

3°. As to the products $\eta_\lambda \eta_\mu$, we have the formulas:

$$\eta_\lambda^2 = \eta_\lambda, \quad \eta_\lambda \eta_\mu = 0 \qquad (\lambda, \mu = 1, \cdots, s;\ \lambda \neq \mu).$$

4°. For each e_k there is one and only one η_λ such that

$$\eta_\lambda e_k = e_k;$$

and for each e_k there is one and only one η_μ such that

$$e_k \eta_\mu = e_k.$$

The particular units η_λ, η_μ thus related to e_k may or may not be distinct. The unit e_k is said to have the *character* (λ, μ).

Every other product of an η with e_k is zero.

5°. The product of two e's contains only e's and, moreover, only e's of lower index than either of the factors—that is, with certain constants γ_{ijs},

* Presented to the Society at the Columbus meeting, August 25, 1899. Received for publication December 6, 1900.

† Mathematische Annalen, vol. 39, pp. 307-313, 1891.

$$\begin{cases} e_i e_j = \sum_{s=1}^{i-1} \gamma_{ijs} e_s & (i \geqq j), \\ e_i e_j = \sum_{s=1}^{j-1} \gamma_{ijs} e_s & (i > j). \end{cases}$$

We note as of especial importance the particular case:

$$e_1 e_i = e_i e_1 = 0 \qquad (i = 1, \cdots, r).$$

6°. (Corollary of 4° and 5°.) If e_i is of character (λ, μ) and e_j of character (μ, ν), the product $e_i e_j$ can contain units of character (λ, ν) only. If e_i is of character (λ, μ) and e_j of character (ρ, σ) where $\rho \neq \mu$, the product $e_i e_j$ is zero.

We shall for the present consider a regular system from which the η's have been deleted, that is, the part consisting of the e's and their products with each other only—this is called a *degenerate system.** A degenerate system is said to be regular if it is in a form to satisfy 5°.

The sequel depends essentially upon the following linear ordering of the products† of the multiplication table.

(*a*) The products are divided into groups and the groups themselves ordered, as follows:

The first group contains all products $e_1 e_j$ and $e_j e_1$ $(j = 1, 2, \cdots, r)$:
The second group contains all products $e_2 e_j$ and $e_j e_2$ $(j = 2, 3, \cdots, r)$:
The k-th group contains all products $e_k e_j$ and $e_j e_k$ $(j = k, k+1, \cdots, r)$:
The $(r-1)$-th group contains all products $e_{r-1} e_j$ and $e_j e_{r-1}$ $(j = r-1, r)$:
The r-th group contains $e_r e_r$.‡

(*b*) Within a group, say the i-th, the products are ordered thus:

$$e_i e_i,\ e_i e_{i+1},\ e_{i+1} e_i,\ e_i e_{i+2},\ e_{i+2} e_i,\ \cdots,\ e_i e_r,\ e_r e_i.$$

In any system § there are certain products, in number m $(0 \leqq m < r)$, each of which is linearly independent of the products preceding it in order. These products, which are of fundamental importance to us, will be called *independent products;* they will be denoted in order of precedence by

$$\pi_1,\ \pi_2,\ \pi_3,\ \cdots,\ \pi_m.$$

* SCHEFFERS's "ausgeartetes System," loc. cit., p. 308.

† "Product" will henceforth be used to denote the product of two e's unless otherwise stated.

‡ If the multiplication table be written in rectangular array, the first group consists of the first line and column, the second group of the part of the second line and column not contained in the first, etc.

§ "System" will mean "degenerate system" while such systems alone are being considered.

To repeat: π_1 is the first product different from zero; π_2 is the first product which cannot be expressed in terms of π_1; π_3 is the first product which cannot be expressed in terms of π_1 and π_2; and so on.

The set of "independent products" is a particular set of linearly independent quadratic products of the system, of which every set then contains m elements. Of course, under a transformation of the system the number m is invariant but the set of "independent products" need not be.

A system whose multiplication table contains m independent products will be called an *m-product* system.

THEOREM I. *Any m-product degenerate non-quaternion number system can be so transformed that* (1) *the transformed system will be regular;* (2) *the m independent products of the transformed system will be the first m units of that system;* (3) *the order of equivalence of independent products and units will be:*

$$\pi_1 = e_1, \ \pi_2 = e_2, \ \cdots, \ \pi_m = e_m.$$

The form indicated by this theorem will be called the *normal form.*

Proof. Assuming that any $(m-1)$-product system can be reduced to the normal form, we shall prove that then any m-product system can be reduced to that form.

Let any regular m-product system be transformed by taking the independent products as the first m units; that is, let

$$\bar{e}_1 = \pi_1, \ \bar{e}_2 = \pi_2, \ \cdots, \ \bar{e}_m = \pi_m,$$

the choice of the remaining new units,

$$\bar{e}_{m+1}, \ \bar{e}_{m+r}, \ \cdots, \ \bar{e}_r,$$

being subject to the usual restrictions in the transformation of a number system, and also to the restriction* that only units of like character shall be united in a transformed unit. We proceed to prove that *the transformed system is regular.*

In the first place, in the transformed system the first m units, and these only, will enter the products. For, the transformed system is an m-product system; hence its products must contain at least m units; again the transformed units are linear combinations of the original units and consequently any product $\bar{e}_i \bar{e}_j$ can be expressed in terms of $\pi_1, \pi_2, \cdots, \pi_m$, that is, in terms of $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_m$. Hence, regularity is preserved in so far as products $\bar{e}_i \bar{e}_j$ $(i > m, j > m)$ are concerned.

It remains to consider the products $\bar{e}_i \bar{e}_j$, where in each product one index is less than or equal to m, that is, the products

$$\pi_i \bar{e}_j, \ \bar{e}_j \pi_i \qquad (i = 1, \cdots, m; j = 1, \cdots, r; i \leqq j)$$

* For the units $\bar{e}_{m+1}, \cdots, \bar{e}_r$ one may indeed choose certain $r - m$ of the original units.

We consider one such pair of products $\pi_i \bar{e}_j$, $\bar{e}_j \pi_i$. In the original system, we have

$$\pi_i = e_s e_t = a_1 e_1 + a_2 e_2 + \cdots + a_{u-1} e_{u-1} \qquad (u \leqq s,\ u \leqq t).$$

Any product in the original system, one of whose factors consists of units of index lower than u precedes the independent product

$$\pi_i = e_s e_t \qquad (u \leqq s,\ u \leqq t),$$

and so, by the definition of independent products, is expressible linearly in terms of $\pi_1, \pi_2, \cdots, \pi_{i-1}$. Hence either product of $\bar{e}_i = \pi_i$ with $\bar{e}_j\ (j \gtrless i)$ can contain only $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{i-1}$. Hence we see that the transformed system is regular.

We now have an m-product regular system in which the units entering products are m in number and are the first m units of the system.* The unit $\bar{e}_1$ being deleted, a regular system remains in which there are $m-1$ units entering products, and $m-1$ independent products. In accordance with our assumption the deleted system may be reduced to the normal form; let the reduction be made and denote the new units by $e_2, e_3 \cdots e_r$, the new independent products by π_1', $\pi_2', \cdots, \pi_{m-1}'$. Since the deleted system is normal,

$$\pi_1' = e_2,\ \pi_2' = e_3,\ \cdots,\ \pi_{m-1}' = e_m.$$

On the restoration of $\bar{e}_1$, which will now be denoted by e_1, we have the original m-product system which we had before deletion but in a new form; it will evidently still be regular.

Since the products $e_1 e_i$, $e_i e_1$ all vanish, if $e_j e_k = e_i$ was an independent product in the deleted system,

$$e_j e_k = e_i + a_i e_1$$

will be an independent product now. Then $m-1$ of the independent products will be

$$e_2 + a_2 e_1,\ e_3 + a_3 e_1,\ \cdots,\ e_m + a_m e_1.$$

Before searching for the remaining independent product let us make the transformation:

$$\bar{e}_2 = e_2 + a_2 e_1,\ \bar{e}_3 = e_3 + a_3 e_1,\ \cdots,\ \bar{e}_m = e_m + a_m e_1,$$

the remaining units being unchanged. The regularity has not been disturbed and the independent products remain unchanged.

Denoting the transformed units by $e_2, e_3, \cdots, e_r$, we have as $m-1$ of the independent products the units $e_2, e_3, \cdots, e_m$. The only other unit which enters any product is e_1; hence, the first product which contains e_1 is the remaining independent product; denote this product $e_g e_h$ by π_t.

* We have by this first transformation made a twofold simplification of the regular form of the system.

If e_i is the independent product immediately preceding π_t,

$$e_g e_h = \pi_t = a_1 e_1 + a_2 e_2 + \cdots + a_i e_i \qquad (a_1 \neq 0;\ i < g,\ i < h).$$

Now we make the transformation:

$$\bar{e}_1 = e_2,\ \bar{e}_2 = e_3,\ \cdots,\ \bar{e}_{i-1} = e_i,\ \bar{e}_i = \pi_t,\ \bar{e}_{i+1} = e_{i+1},\ \cdots,\ \bar{e}_r = e_r,$$

and prove that the transformed system will be regular and that its independent products will be

$$\pi_1 = \bar{e}_1,\ \pi_2 = \bar{e}_2,\ \cdots,\ \pi_{m-1} = \bar{e}_{m-1},\ \pi_m = \bar{e}_m,$$

that is, that the transformed system is in the normal form.

The transformation in effect omits the unit e_1 and leaving the remaining units in their original order introduces π_t as the new unit $\bar{e}_i$. The products of the new system are the products of the old (apart from the vanishing e_1-products) and the products involving $\pi_t = \bar{e}_i$. Of these all not involving $\bar{e}_i$ obviously have regular expressions, since in the original system the first product involving e_1 was $e_g e_h = \pi_t = \bar{e}_i$ $(g > i,\ h > i)$, and so all products $\bar{e}_k \bar{e}_l$ involving e_1 have $k > i$, $l > i$ and may be expressed regularly, the e_1 being replaced by $\bar{e}_i$, $\bar{e}_1$, $\cdots$, $\bar{e}_{i-1}$. And it is easy to see also that the products involving $\bar{e}_i$ are expressible regularly.

Further, in the new system $\bar{e}_1$, $\cdots$, $\bar{e}_{i-1}$ arise as before as sequential independent products from the products preceding the product $e_g e_h$, the products involving $\bar{e}_i$ as a factor obviously being expressible in terms of the products involving $\bar{e}_1, \cdots, \bar{e}_{i-1}$. Then, as the next independent product,

$$\pi_t = e_g e_h = \bar{e}_i = \bar{e}_g \bar{e}_h \qquad (i < g,\ i < h)$$

enters, and by the intervention of the $\bar{e}_i$, with $\bar{e}_1$, $\cdots$, $\bar{e}_{i-1}$ to replace the original e_1, the $\bar{e}_{i+1}, \cdots, \bar{e}_m$ enter as before as sequential independent products. Thus after the transformation the independent products in order are $\bar{e}_1, \cdots, \bar{e}_m$.

Thus the general step of the induction proof is established, and it is evident that a zero-product system is normal; hence we have proved that *every system can be brought into the normal form.*

THEOREM II. *The product of two independent products is not an independent product.*

Let e_i and e_j $(i < j)$ be independent products in a normal system and let $e_p e_q = e_j$.

Then

$$e_i e_j = (e_i e_p) e_q = (a_1 e_1 + a_2 e_2 + \cdots + a_{i-1} e_{i-1}) e_q.$$

Hence $e_i e_j$ is equal to a sum of products which precede it, and cannot be an independent product.

We have thus far considered the degenerate system, and have defined the normal form for that alone. We call the complete non-quaternion system normal if the system is regular and the degenerate system normal. It is necessary to show that the transformations employed have not destroyed regularity with respect to the products of the e's with the η's in order to extend our results to the complete system.

The regularity of the complete system could be destroyed by transformations of the degenerate system which maintain its own regularity in one way only; that is by uniting in a transformed unit, two or more units of different characters.

In the first transformation of the degenerate system it was explicitly stated that the units united in a transformed unit should all be of the same character.

The other transformations which united two or more units in a transformed unit were made by means of equations of the form:

$$\bar{e}_i = e_j e_k,$$

and all such transformations are permissible, for one sees by 6° that the units contained in a product $e_i e_j$ must all be of the same character. Hence we have:

THEOREM III. *Any non-quaternion system may be reduced to the normal form.*

YALE UNIVERSITY, *June*, 1900.

ON THE REDUCTION OF THE GENERAL ABELIAN INTEGRAL*

BY

J. C. FIELDS

§ 1.

In their exceedingly interesting book *Théorie des Fonctions algébriques et de leurs Intégrales*, pp. 344–345, APPELL and GOURSAT have given a brief sketch of HERMITE'S method for obtaining by rational operations the reduced form for a hyperelliptic integral, prefacing the same by a remark which would seem to imply that the more general problem in the case of the Abelian integrals was still awaiting its solution.

To this remark was due an attempt on my part to solve the indicated problem, and the method by which I succeeded in effecting its solution will be found in the present paper. The results here given, it may be remarked, have been in my possession now for about four years and were presented at the annual meeting of the American Mathematical Society held in Toronto in 1897 but have not hitherto been published.

Let

$$F(z, u) = \sum e_{r,s} z^r u^s = 0 \tag{1}$$

be the equation to an irreducible algebraic curve of degree n.

The form of this equation we shall suppose to be the standard one in which the only multiple points are double points separate from the branch points and with distinct tangents. We shall throughout regard u as the dependent variable and shall further assume that no line parallel to the axis of u is an inflexional tangent or includes more than one point of the nature of a branch or double point or is tangent at a double point, the singular points too being all supposed to be finite and the asymptotes all distinct and none of them parallel to an axis and no two parallel to each other.

That the most general algebraic equation can be reduced to such a form by a birational transformation is a well-known fact, and in such a reduced form it is customary to treat the equation.

* Presented to the Society at the Toronto meeting, August 16–17, 1897, under a different title. Received for publication November 28, 1900.

On the above hypotheses a term in u^n will present itself in the equation to our curve and we shall find it convenient to assume that its coefficient e_{0n} has the value 1.

Also as a consequence of the said hypotheses z and u can only become infinite together, and further for any given value $z = a$ the equation $F(a, u) = 0$ will never furnish less than $n - 1$ distinct values of u.

On indicating any arbitrary rational function of z, u by $R(z, u)$ the general Abelian integral corresponding to our curve $F(z, u) = 0$ will be

$$\int R(z, u)dz. \tag{2}$$

This integral we know may be expressed as the sum of a rational function of z, u, of an integral of the first kind, of a linear function with constant coefficients of p elementary integrals of the second kind, and of a number of elementary integrals of the third kind, where

$$p = \tfrac{1}{2}(n-1)(n-2) - d$$

is the number known as the "deficiency" of the curve, d indicating the number of the double points.

Starting out now with any arbitrary Abelian integral we purpose in this paper to reduce the same to a sum of the character in question, employing thereto a succession of processes, none of which involves other than rational operations in our variables.

By aid of the relation $F(z, u) = 0$ it is known that any rational function $R(z, u)$ may be reduced to the form of a polynomial in u of the degree $n - 1$, whose coefficients are rational functions of z, and on applying the principle of partial fractions to these functions of z we may express $R(z, u)$ in the form:

$$R(z, u) = \sum_{\lambda} R_{\lambda}\left(\frac{1}{z - a_{\lambda}}, u\right) + R_0(z, u), \tag{3}$$

where $R_{\lambda}(1/(z - a_{\lambda}), u)$ denotes a polynomial in $1/(z - a_{\lambda})$, u and $R_0(z, u)$ a polynomial in z, u, the degrees of these polynomials in the variable u not being greater than $n - 1$.

Our Abelian integral then takes the form:

$$\int R(z, u)dz = \sum_{\lambda} \int R_{\lambda}\left(\frac{1}{z - a_{\lambda}}, u\right)dz + \int R_0(z, u)dz. \tag{4}$$

We shall assume that $R_{\lambda}(1/(z - a_{\lambda}), u)$ contains only negative powers of $z - a_{\lambda}$, as all terms involving other than such powers may be supposed to be included under $R_0(z, u)$.

§ 2.

For the further reduction of our integral we have need of a formula which we shall now derive. The formula in question is the following:

$$\frac{du}{dz} = -\frac{F'_z}{F'_u} = \sum_e \frac{\phi_e(u)}{z - a_e} = \sum_e \frac{c_e F(a_e, u)}{(z - a_e)(u - b_e)}, \tag{5}$$

where the summation is supposed to extend to all branch and double points (a_e, b_e), and where for the constant coefficients c we have

$$c_e = \left[\frac{1}{F''_u(a_e, u)}\right]_{u=b_e} \text{ or } \left[\frac{2}{F''_u(a_e, u)}\right]_{u=b_e}$$

according as (a_e, b_e) is a branch or double point.

Starting out from the identity

$$-\frac{F'_z}{F'_u} = -\frac{F'_z[F'_u]_{u=u_1} \cdots [F'_u]_{u=u_{n-1}}}{F'_u[F'_u]_{u=u_1} \cdots [F'_u]_{u=u_{n-1}}},$$

where $u_1, u_2, \cdots, u_{n-1}$ are the $n-1$ roots of $F(z, u) = 0$ conjugate to u, we see that the denominator of the expression on the right is a symmetric function of the n quantities $u, u_1, \cdots, u_{n-1}$ and therefore a rational function of z. It is, in fact, the discriminant, an integral rational function of z of degree $n(n-1)$ whose factors $z - a_e$ correspond to the several branch and double points. In case a double point corresponds to the value $z = a_e$ the factor $z - a_e$ will appear to the second power in the discriminant and in the case of a branch point to the first power, as is well known.

The factor $[F'_u]_{u=u_1} \cdots [F'_u]_{u=u_{n-1}}$ in the numerator of the above expression is symmetric in the $n-1$ quantities $u_1 \cdots u_{n-1}$ and may evidently be expressed as a polynomial in z, u. The numerator itself may then be written in the form:

$$A_1 u^{n-1} + A_2 u^{n-2} + \cdots + A_n,$$

where the coefficients A are polynomials in z.

On indicating the discriminant by the letter D we then obtain for du/dz the expression

$$\frac{du}{dz} = \frac{A_1 u^{n-1} + A_2 u^{n-2} + \cdots + A_n}{D}, \tag{6}$$

where, as may readily be shown, the degree of the numerator in z, u is not greater than $n(n-1)$ and where in particular the degree of the term A_n is less than this number.

Suppose, namely, that the directions of the n points at ∞ on the curve $F(z, u) = 0$ are given by the equations

$$u - \kappa_1 z = 0\,, \quad u - \kappa_2 z = 0, \cdots, u - \kappa_n z = 0\,, \tag{7}$$

where, according to our original hypothesis in regard to the asymptotes of our curve, the n coefficients κ all have different values. The differential coefficient du/dz will then have the values $\kappa_1, \kappa_2, \cdots, \kappa_n$, respectively, for the several branches at ∞.

Indicating the degree of the numerator in the expression on the right of (6) by ν and representing by $a_1, a_2, \cdots, a_n$ the coefficients of the terms of this degree in the elements $A_1 u^{n-1}, A_2 u^{n-2}, \cdots, A_n$ respectively, we shall have

$$a_1 \kappa_r^{n-1} + a_2 \kappa_r^{n-2} + \cdots + a_n \tag{8}$$

for the coefficient of z^r in the numerator in question, on substituting for u in terms of z from the rth branch of the curve. Now in case we had $\nu > n(n-1)$ the differential coefficient in (6) would be infinite for some branch of the curve at ∞ unless the expression (8) be equal to 0 for each of the n values $\kappa_1, \kappa_2, \cdots, \kappa_n$ of κ_r. In this case, however, the expression in question being only of the degree $n-1$ must vanish identically and we should have $a_1 = a_2 = \cdots = a_n = 0$. It follows therefore that ν cannot be greater than $n(n-1)$.

Assuming then that we have $\nu = n(n-1)$, substituting for u in terms of z from the equation to one of the n branches at ∞, and putting $z = \infty$ in (6), we derive

$$\kappa_r = \frac{a_1}{\delta} \kappa_r^{n-1} + \frac{a_2}{\delta} \kappa_r^{n-2} + \cdots + \frac{a_n}{\delta}\,, \tag{9}$$

where δ is the coefficient of $z^{n(n-1)}$ in D.

Now this equation is of the degree $n-1$ in κ_r and must be satisfied by the n quantities $\kappa_1, \kappa_2, \cdots, \kappa_n$. It must therefore be an identity, and we consequently have

$$a_1 = 0\,, \; a_2 = 0\,, \; \cdots, \; a_{n-2} = 0\,, \; a_{n-1} = \delta\,, \; a_n = 0\,.$$

We see therefore that the degree of the coefficient A_n, as well as that of the other coefficients A in the numerator of the expression to the right of (6), must be less than $n(n-1)$. On applying to this expression the principle of partial fractions, the various parts will evidently all be fractional in their character.

The expression in (6) may then be written in the form:

$$\frac{du}{dz} = \sum_c \left\{ \frac{\phi_c(u)}{z - a_c} + \frac{\psi_c(u)}{(z - a_c)^2} \right\}, \tag{10}$$

where the summation extends to all those values $z = a_c$ to which correspond branch or double points, and where the numerators $\phi(u)$ and $\psi(u)$ are polynomials in u of degree $n-1$.

We have already remarked that $z - a_c$ is a quadratic factor of the discriminant D in case a double point corresponds to the value $z = a_c$, whereas in the case of a branch point the corresponding factor enters only to the first power.

In the summation above, then, the numerator $\psi_\epsilon(u)$ in an element corresponding to a branch point must evidently be 0, and we shall prove that this also is so in the case of an element corresponding to a double point.

Consider, namely, an element

$$\frac{\phi(u)}{z-a}+\frac{\psi(u)}{(z-a)^2},$$

where to $z=a$ corresponds a double point. The other points of our curve corresponding to this value of the variable will then be ordinary points by virtue of our original hypothesis, and the differential coefficient $du\,dz$ will evidently be finite for each of the $n-1$ points corresponding to the value $z=a$. Its values for the two branches at the double point I shall indicate by m_1 and m_2 respectively.

For the $n-1$ points corresponding to the value $z=a$ the summation in (10) and therewith the element here in question must be finite. It evidently follows that $\psi(u)$ must be equal to 0 for each of the $n-1$ values of u corresponding to $z=a$, and on writing

$$F(a,\ u)=(u-b)^2(u-b')(u-b'')\cdots(u-b^{(n-2)}),$$

we must have

$$\psi(u)=c(u-b)(u-b')\cdots(u-b^{(n-2)}),$$

where c is a constant, for $\psi(u)$ has the degree $n-1$.

We may therefore write

$$\psi(u)=\frac{cF(a,\ u)}{u-b},$$

and our element will have the form

$$\frac{\phi(u)}{z-a}+\frac{\psi(u)}{(z-a)^2}=\frac{1}{z-a}\left\{\phi(u)+\frac{cF(a,\ u)}{(z-a)(u-b)}\right\}.$$

Now in order that this element may be finite for the two branches at the double point it is necessary that the expression in parentheses should vanish for both these branches.

Writing this expression in the form

$$\phi(u)+\frac{u-b}{z-a}\frac{cF(a,\ u)}{(u-b)^2},$$

we see then that we must have

$$\phi(b)+m_1\left[\frac{cF(a,\ u)}{(u-b)^2}\right]_{u=b}=0,$$

$$\phi(b)+m_2\left[\frac{cF(a,\ u)}{(u-b)^2}\right]_{u=b}=0.$$

From this it follows that

$$c(m_1 - m_2)\left[\frac{F(a, u)}{(u-b)^2}\right]_{u=b} = 0,$$

and as we have

$$m_1 \neq m_2, \quad \left[\frac{F(a, u)}{(u-b)^2}\right]_{u=b} \neq 0,$$

we must have $c = 0$.

It follows therefore that $\psi(u) = 0$ and the summation in (10) reduces to the form

$$\frac{du}{dz} = \sum_e \frac{\phi_e(u)}{z - a_e},$$

where, as before, the summation is supposed to extend to every value $z = a_e$ to which corresponds a branch or double point. Now the differential coefficient du/dz can only become infinite for the branch points, and for a branch point too it can only become infinite like $(z-a)^{-\frac{1}{2}}$, so that we see that in any case the numerator $\phi_e(u)$ of an element in the above summation must equal 0 for each of the $n-1$ values of u corresponding to the value $z = a_e$. It follows therefrom that $\phi_e(u)$ must have the form

$$\phi_e(u) = \frac{c_e F(a_e, u)}{u - b_e},$$

and for the differential coefficient du/dz we obtain the expression

$$\frac{du}{dz} = \sum_e \frac{c_e F(a_e, u)}{(z-a_e)(u-b_e)},$$

where it only remains now to determine the value of the constants c_e.

First considering the case of a double point (a_e, b_e) we may evidently write the expression for our differential coefficient in the form

$$\frac{du}{dz} = \frac{c_e F(a_e, u)}{(u-b_e)^2} \cdot \frac{u - b_e}{z - a_e} + f(z, u),$$

where the value of the function $f(z, u)$ for the point (a_e, b_e) is unambiguous.

On considering the respective values of this expression for the two branches passing through the double point, we shall have

$$m_1 = \left[\frac{c_e F(a_e, u)}{(u-b_e)^2}\right]_{u=b_e} m_1 + f(a_e, b_e) = \tfrac{1}{2} c_e m_1 [F'''_u]_{a_e b_e} + f(a_e, b_e),$$

$$m_2 = \left[\frac{c_e F(a_e, u)}{(u-b_e)^2}\right]_{u=b_e} m_2 + f(a_e, b_e) = \tfrac{1}{2} c_e m_2 [F'''_u]_{a_e b_e} + f(a_e, b_e);$$

whence by subtraction

$$(m_1 - m_2) = \tfrac{1}{2}(m_1 - m_2) c_e [F'''_u]_{a_e b_e},$$

and therefore

$$c_e = \frac{2}{[F'''_u]_{a_e b_e}}.$$

In the case of a branch point (a_e, b_e) we evidently have

$$\left[\frac{(u-b_e)^2}{z-a_e}\right]_{a_e b_e} = \left[\frac{d}{dz}(u-b_e)^2\right]_{a_e b_e} = \left[2(u-b_e)\frac{du}{dz}\right]_{a_e b_e}.$$

We have however also:

$$\left[(u-b_e)\frac{du}{dz}\right]_{a_e b_e} = \left[\frac{c_e F(a_e, u)}{z-a_e}\right]_{a_e b_e} = \left[\frac{c_e F(a_e, u)}{(u-b_e)^2}\frac{(u-b_e)^2}{z-a_e}\right]_{a_e b_e};$$

and from these two formulæ combined we derive:

$$\left[\frac{(u-b_e)^2}{z-a_e}\right]_{a_e b_e} = \left[\frac{2c_e F(a_e, u)}{(u-b_e)^2}\frac{(u-b_e)^2}{z-a_e}\right]_{a_e b_e}.$$

Here the expression to the left is evidently finite, and it is also different from 0 since by our original hypothesis the branch point (a_e, b_e) is not at the same time a double point. We may therefore divide both sides of our equation by the expression in question, and obtain

$$1 = c_e\left[\frac{2F(a_e, u)}{(u-b_e)^2}\right]_{a_e b_e} = c_e [F'''_u]_{a_e b_e};$$

whence in the case of a branch point (a_e, b_e) we have

$$c_e = \frac{1}{[F'''_u]_{a_e b_e}}.$$

We have therefore established our formula (5).

§ 3.

Returning now to the reduction of our integral in (4) we shall first consider an individual element,

$$\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right) dz,$$

in the summation appearing on the right.

Supposing the highest exponent of $1/(z-a_\lambda)$ in the expression under the integral sign to be greater than 1 and indicating the same by r we may write:

$$R_\lambda\left(\frac{1}{z-a_\lambda}, u\right) = \frac{f_r(u)}{(z-a_\lambda)^r} + R'\left(\frac{1}{z-a_\lambda}, u\right),$$

where $f(u)$ is a polynomial in u of degree $n-1$ and where $R'(1/(z-a_\lambda), u)$ is a polynomial in $1/(z-a_\lambda)$, u of degrees $r-1$ and $n-1$ respectively in $1/(z-a_\lambda)$ and u.

We shall then have:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz = \int \frac{f_r(u)}{(z-a_\lambda)^r}dz + \int R'\left(\frac{1}{z-a_\lambda}, u\right)dz$$

$$= \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \int\left\{\frac{f_r(u)}{(z-a_\lambda)^r} - \frac{d}{dz}\frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}}\right\}dz + \int R'\left(\frac{1}{z-a_\lambda}, u\right)dz$$

(11)

$$= \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \int\left\{\frac{f_r(u)+(r-1)g_{r-1}(u)}{(z-a_\lambda)^r} - \sum_e \frac{g'_{r-1}(u)\cdot\phi_e(u)}{(z-a_\lambda)^{r-1}(z-a_e)}\right\}dz + \int R'\left(\frac{1}{z-a_\lambda}, u\right)dz;$$

where $g_{r-1}(u)$ is a polynomial in u of degree $n-1$, as yet undetermined, and where for du/dz we have substituted its expression as given in (5), so that the summation with regard to e which here appears is supposed to extend to all branch and double points. We shall first suppose that none of the points of the curve corresponding to the value $z = a_\lambda$ is a branch or double point. In this case then we shall have $a_\lambda \neq a_e$ for all values of the a_e's. For the polynomial $g_{r-1}(u)$ then choosing

$$g_{r-1}(u) = -\frac{f_r(u)}{r-1},$$

we reduce our integral (11) to the form:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz = \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} - \int\left\{\sum_e \frac{g'_{r-1}(u)\cdot\phi_e(u)}{(z-a_\lambda)^{r-1}(z-a_e)}\right\}dz + \int R'\left(\frac{1}{z-a_\lambda}, u\right)dz.$$

Here, in the expression to the right, $(z-a_\lambda)^{r-1}$ is the highest power of $z-a_\lambda$ which appears in a denominator under the integral sign, and on reducing the order to which u appears in the products $g'_{r-1}(u)\cdot\phi_e(u)$ by aid of the equation $F(z, u) = 0$ and on separating the factors $(z-a_\lambda)^{r-1}$ and $z-a_e$ from one another in the denominators by the method of partial fractions, we express our integral in the form:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz = \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \int R'_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz + \sum_e \int S'_e\left(\frac{1}{z-a_e}, u\right)dz + \int S'_0(z, u)dz,$$

(12)

where $R'_\lambda(1/(z-a_\lambda),\ u)$ is a polynomial in $1/(z-a_\lambda)$, u of degree $r-1$ in $1/(z-a_\lambda)$, $S'_e(1/(z-a_e),\ u)$ a polynomial in $1/(z-a_e)$, u of degree 1 in $1/(z-a_e)$, and $S'_0(z,\ u)$ a polynomial in z, u, none of these polynomials having in u a degree which is greater than $n-1$.

By the above process we have reduced by 1 the degree to which $1/(z-a_\lambda)$ appears under the integral sign and by application of the same process to

$$\int R'_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz$$

we should obtain a formula:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz=\frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}}+\frac{g_{r-2}(u)}{(z-a_\lambda)^{r-2}}+\int R''_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz$$
$$+\sum_e\int S''_e\left(\frac{1}{z-a_e},\ u\right)dz+\int S''_0(z,\ u)dz,$$

where $r-2$ is the degree of $R''_\lambda(1/(z-a_\lambda),\ u)$ in $1/(z-a_\lambda)$.

By successive applications of the process we should ultimately arrive at an expression for our integral in the form:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz=\frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}}+\frac{g_{r-2}(u)}{(z-a_\lambda)^{r-2}}+\cdots+\frac{g_1(u)}{z-a_\lambda}$$
$$+\int \bar{R}_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz+\sum_e\int \bar{S}_e\left(\frac{1}{z-a_e},\ u\right)dz+\int \bar{S}_0(z,\ u)dz,\tag{13}$$

where $\bar{R}_\lambda(1/(z-a_\lambda),\ u)$ is a polynomial in $1/(z-a_\lambda)$, u of degree 1 in $1/(z-a_\lambda)$, $\bar{S}_e(1/(z-a_e),\ u)$ a polynomial in $1/(z-a_e)$, u of degree 1 in $1/(z-a_e)$, and $\bar{S}_0(z,\ u)$ a polynomial in z, u.

§ 4.

We have obtained formula (13) on the assumption that no branch or double point corresponds to the value $z=a_\lambda$. Assuming on the contrary now that we have a branch or double point $(a_\lambda,\ b_\lambda)$ so that $z-a_\lambda$ is included among the factors $z-a_e$, we derive from (11):

$$\int R_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz=\frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}}+\int\left\{\frac{f_r(u)+(r-1)g_{r-1}(u)-g'_{r-1}(u)\cdot\phi_\lambda(u)}{(z-a_\lambda)^r}\right.$$
$$\left.-\sum_e{}'\frac{g'_{r-1}(u)\cdot\phi_e(u)}{(z-a_\lambda)^{r-1}(z-a_e)}\right\}dz+\int R'\left(\frac{1}{z-a_\lambda},\ u\right)dz,\tag{14}$$

where the accent in the sign of summation $\sum'_e$ is intended to signify that the term corresponding to $a_e=a_\lambda$ is omitted from the summation.

Here again we wish by a proper choice of $g_{r-1}(u)$ to rid ourselves of terms under the integral sign which involve $(z-a_\lambda)^r$ in their denominator.

For this it will be necessary that the expression

$$f_r(u) + (r-1)g_{r-1}(u) - g'_{r-1}(u)\cdot\phi_\lambda(u) \tag{15}$$

be divisible by $z-a_\lambda$, after it has been reduced to the normal form in which u does not appear to an order higher than $n-1$. Now if this expression in its reduced form is to be divisible by $z-a_\lambda$ it must, as it stands, be equal to 0 for each of the $n-1$ values $b_\lambda, b'_\lambda, \cdots, b_\lambda^{(n-2)}$ of u corresponding to the value $z=a_\lambda$, and must therefore be divisible by

$$\phi_\lambda(u) = c_\lambda(u-b_\lambda)(u-b'_\lambda)\cdots(u-b_\lambda^{(n-2)}) = \frac{c_\lambda F(a_\lambda, u)}{u-b_\lambda}.$$

It follows then that

$$f_r(u) + (r-1)g_{r-1}(u)$$

also must be divisible by $\phi_\lambda(u)$, and since the degree of this expression in u is $n-1$ we may write:

$$f_r(u) + (r-1)g_{r-1}(u) = \kappa\phi_\lambda(u),$$

where κ is a constant which is yet to be determined.

For our function $g_{r-1}(u)$ we must therefore have

$$g_{r-1}(u) = \frac{\kappa\phi_\lambda(u) - f_r(u)}{r-1}, \tag{16}$$

and on substituting this in (15) we obtain:

$$\begin{aligned} f_r(u) + (r-1)g_{r-1}(u) - g'_{r-1}(u)\cdot\phi_\lambda(u) &= \big(\kappa - g'_{r-1}(u)\big)\phi_\lambda(u) \\ &= \frac{1}{r-1}\{(r-1)\kappa - \kappa\phi'_\lambda(u) + f'_r(u)\}\phi_\lambda(u), \end{aligned} \tag{17}$$

which expression in its reduced form must be divisible by $z-a_\lambda$. Now we have:

$$(u-b_\lambda)\phi_\lambda(u) = c_\lambda F(a_\lambda, u) = c_\lambda\{F(a_\lambda, u) - F(z, u)\},$$

which is evidently divisible by $z-a_\lambda$. Also in order that the expression in (17) should be divisible by $(u-b_\lambda)\phi_\lambda(u)$ it is only necessary that the factor

$$(r-1)\kappa - \kappa\phi'_\lambda(u) + f'_r(u)$$

should be divisible by $u-b_\lambda$, which will be the case if we choose κ so as to satisfy the condition

$$\{(r-1) - \phi'_\lambda(b_\lambda)\}\kappa + f'_r(b_\lambda) = 0. \tag{18}$$

On so choosing κ the expression in (17) may be put into a form which is divisible by $z-a_\lambda$ and must therefore evidently also, in its reduced form, be

divisible by this factor. Such a choice of κ is of course always possible, excepting in the case where the coefficient of κ in (18) is equal to 0.

We have:

$$\phi'_\lambda(b_\lambda) = c_\lambda[(u - b'_\lambda)\cdots(u - b_\lambda^{(n-2)})]_{u=b_\lambda} = \tfrac{1}{2}c_\lambda[F'''_u(a_\lambda, u)]_{u=b_\lambda}.$$

It follows then that

$$\phi'_\lambda(b_\lambda) = \tfrac{1}{2} \quad \text{or} \quad 1,$$

and that we therefore have:

$$\kappa = -\frac{f'_r(b_\lambda)}{r - 1 - \phi'_\lambda(b_\lambda)} = \frac{-f'_r(b_\lambda)}{r - \frac{3}{2}} \quad \text{or} \quad \frac{-f'_r(b_\lambda)}{r - 2},$$

according as (a_λ, b_λ) is a branch or double point. The one case here excepted is evidently that in which we have to do with a double point (a_λ, b_λ) and where at the same time $r = 2$, in which case the coefficient of κ in (18) is 0. Substituting the values just obtained for κ in the expression already given above for $g_{r-1}(u)$ we obtain:

$$g_{r-1}(u) = -\left\{\frac{4f'_r(b_\lambda)\cdot\phi_\lambda(u)}{(2r-2)(2r-3)} + \frac{f_r(u)}{r-1}\right\}$$

$$= -\left\{\frac{4}{(2r-2)(2r-3)}\frac{f'_r(b_\lambda)}{F'''_{b_\lambda}(a_\lambda, b_\lambda)}\frac{F(a_\lambda, u)}{u - b_\lambda} + \frac{f_r(u)}{r-1}\right\} \quad \text{or}$$

(19)

$$g_{r-1}(u) = -\left\{\frac{f'_r(b_\lambda)\cdot\phi_\lambda(u)}{(r-1)(r-2)} + \frac{f_r(u)}{r-1}\right\}$$

$$= -\left\{\frac{2}{(r-1)(r-2)}\frac{f'_r(b_\lambda)}{F'''_{b_\lambda}(a_\lambda, b_\lambda)}\frac{F(a_\lambda, u)}{u - b_\lambda} + \frac{f_r(u)}{r-1}\right\},$$

according as (a_λ, b_λ) is a branch point or a double point, the exceptional case being here of course excluded.

The substitution of these expressions for $g_{r-1}(u)$ in (14) will enable us to rid ourselves of the r-th power of $z - a_\lambda$ in the denominator under the integral sign, the numerator

$$f_r(u) + (r-1)g_{r-1}(u) - g'_{r-1}(u)\cdot\phi_\lambda(u)$$

in its reduced form being then divisible by $z - a_\lambda$.

By applying the principle of partial fractions to the terms of form

$$\frac{g'_{r-1}(u)\cdot\phi_e(u)}{(z - a_\lambda)^{r-1}(z - a_e)},$$

we may, as in the section preceding, separate the factors $(z - a_\lambda)^{r-1}$ and $z - a_e$ in the denominators, and on reducing the powers of u occurring in the various numerators we may put (14) into the form:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz = \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \int R'_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz + \sum_e{}' \int S'_e\left(\frac{1}{z-a_e}, u\right)dz + \int S'_0(z, u)\,dz,$$

where $R'_\lambda(1/(z-a_\lambda), u)$ is a polynomial in $1/(z-a_\lambda)$, u of degree $r-1$ in $1/(z-a_\lambda)$, $S'_e(1/(z-a_e), u)$ a polynomial in $1/(z-a_e)$, u of degree 1 in $1/(z-a_e)$, and $S'_0(z, u)$ a polynomial in z, u, the degrees in u of these several polynomials being not greater than $n-1$.

By successive applications of the above process we shall evidently arrive at a reduced form,

$$\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz = \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \frac{g_{r-2}(u)}{(z-a_\lambda)^{r-2}} + \cdots + \frac{g_1(u)}{z-a_\lambda} + \int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz + \sum_e \int \bar{S}_e\left(\frac{1}{z-a_e}, u\right)dz + \int \bar{S}_0(z, u)dz, \tag{20}$$

which is a form of the same type as that obtained in (13) for the case of the section preceding, excepting that here $1/(z-a_\lambda)$ may present itself to the second power in the polynomial $R_\lambda(1/(z-a_\lambda), u)$ where to $z=a_\lambda$ corresponds a double point.

We shall now consider more in detail the special case $r=2$ in which we have failed to get rid of the power $1/(z-a_\lambda)^2$ under the sign of integration.

The difficulty here arose in connection with the equation (18) which takes the form

$$0\cdot\kappa + f'_2(b_\lambda) = 0,$$

when $r=2$ and when at the same time (a_λ, b_λ) is a double point. This equation cannot be satisfied by any finite value of κ unless we happen to have $f'_2(b_\lambda)=0$, in which case κ may have any arbitrary value. Supposing then that we have to do with this case we shall take $\kappa=0$ when from (16) we obtain

$$g_{r-1}(u) = -\frac{f'_r(u)}{r-1},$$

and the expression in (17) becomes

$$\frac{1}{r-1}f'_r(u)\cdot\phi_\lambda(u),$$

which is divisible by $(u-b_\lambda)\phi_\lambda(u)$ and in its reduced form is therefore divisible by $z-a_\lambda$.

The numerator of $(z-a_\lambda)^r = (z-a_\lambda)^2$ in (14) is then divisible by $z-a_\lambda$ and $1/(z-a_\lambda)$ will therefore appear only to the first power under the sign of integration.

If however we have $f_2'(b_\lambda) \neq 0$, I shall define a function $\bar{f}_2(u)$ by the equation

$$f_2(u) = \bar{f}_2(u) + \bar{h}\phi_\lambda(u),$$

where $\bar{h} = f_2'(b_\lambda)$.

Since $\phi_\lambda'(b_\lambda) = 1$, it follows then that $\bar{f}_2'(b_\lambda) = 0$.

On writing $r = 2$ in (11) we shall have:

$$\begin{aligned}\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz &= \int \frac{f_2(u)}{(z-a_\lambda)^2}dz + \int R'\left(\frac{1}{z-a_\lambda}, u\right)dz \\ &= \int \frac{\bar{f}_2(u)}{(z-a_\lambda)^2}dz + \int \frac{\bar{h}\phi_\lambda(u)}{(z-a_\lambda)^2}dz + \int R'\left(\frac{1}{z-a_\lambda}, u\right)dz \\ &= \int \frac{\bar{f}_2(u)}{(z-a_\lambda)^2}dz + \int \frac{hF(a_\lambda, u)}{(z-a_\lambda)^2(u-b_\lambda)}dz \\ &\qquad + \int R'\left(\frac{1}{z-a_\lambda}, u\right)dz.\end{aligned}$$

Apart from the second integral on the right the expression here appearing is of the type just considered, in which we saw that we could get rid of the second power of $1/(z-a_\lambda)$ under the sign of integration.

Hence in the case where to $z = a_\lambda$ corresponds a double point we may evidently replace (20) by a formula of the type:

$$\begin{aligned}\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz &= \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \frac{g_{r-2}(u)}{(z-a_\lambda)^{r-2}} + \cdots + \frac{g_1(u)}{z-a_\lambda} \\ &+ h_\lambda \int \frac{F(a_\lambda, u)}{(z-a_\lambda)^2(u-b_\lambda)}dz + \sum_e \int \bar{S}_e\left(\frac{1}{z-a_e}, u\right)dz + \int \bar{S}_0(z, u)dz;\end{aligned} \tag{21}$$

and under this formula, on putting 0 for the constant coefficient h_λ, may be supposed to be included that also for the case in which a branch point corresponds to $z = a_\lambda$, for in this case the integral

$$\int \bar{R}_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz$$

in (20) may be supposed to be included under the summation

$$\sum_e \int \bar{S}\left(\frac{1}{z-a_e}, u\right)dz,$$

the first power only of $1/(z-a_\lambda)$ here presenting itself under the sign of integration.

§ 5.

By the processes of the two preceding sections, therefore, we can reduce any integral

$$\int R_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz$$

to the form given on the right of (21) or (13) respectively, according as a singular point does or does not correspond to the value $z=a_\lambda$. After the substitution of the reduced forms so obtained for the several integrals in the summation to the right of (4) this equation will assume the form:

$$\int R(z,\ u)dz = P(z,\ u) + \sum_\lambda \int \frac{h_\lambda F(a_\lambda,\ u)}{(z-a_\lambda)^2(u-b_\lambda)}\,dz + \sum_\lambda \int \bar{R}_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz + \sum_e \int S_e''\left(\frac{1}{z-a_e},\ u\right)dz + \int S_0''(z,\ u)dz,$$

where the first summation on the right-hand side is supposed to extend to the double points $(a_\lambda,\ b_\lambda)$ only, the second summation to those values of a_λ for which negative powers of $z-a_\lambda$ occur in $R(z,\ u)$, from which however we may exclude elements corresponding to branch or double points, and the third summation to those values a_e to which correspond branch or double points.

The last two summations we may fuse into one, reducing the above formula to the form:

$$\int R(z,\ u)dz = P(z,\ u) + \sum_\lambda \int \frac{h_\lambda F(a_\lambda,\ u)}{(z-a_\lambda)^2(u-b_\lambda)}\,dz + \sum_\lambda \int S_\lambda''\left(\frac{1}{z-a_\lambda},\ u\right)dz + \int S_0''(z,\ u)dz, \tag{22}$$

where the first summation on the right still extends to the double points alone while the second summation extends to those values of a_λ to which correspond branch or double points and over and above these to all those values for which negative powers of the corresponding factor $z-a_\lambda$ appear in the expression of the function $R(z,\ u)$.

The function

$$\frac{F(a_\lambda,\ u)}{(z-a_\lambda)^2(u-b_\lambda)}$$

becomes infinite to the first order for each of the $n-1$ points corresponding to the value $z=a_\lambda$ and it is evident in fact that for a finite value of z none of the functions under the sign of integration to the right of (22) becomes infinite to an order greater than 1.* Apart from the points at ∞, therefore, the integrals

* While employing the form of expression "infinity of the first, second, etc., order" in the accepted sense of RIEMANN, I shall find it convenient for the temporary purpose of this paper to

on the right of (22) can only become infinite logarithmically. The residues corresponding to the several infinities are immediately determinable and we may suppose the logarithmic portion of the expression in (22) to be represented by an integral of the third kind which we shall indicate by (III) and which is made up of a sum with constant coefficients of a number of elementary integrals of the third kind.

Formula (22) may then be reduced to the form

$$\int R(z, u)dz = P(z, u) + \text{III} + \int H(z, u)dz, \tag{23}$$

where the integral $\int H(z, u)dz$ does not become infinite for any finite value of z. The function $H(z, u)$ itself will then not become infinite to an order as great as 1 for any finite value of the variable. What the form of our function $H(z, u)$ is would appear immediately from the forms of the expressions under the signs of integration to the right of (22).

We shall find it convenient however to here determine directly once for all the general form of a rational function $H(z, u)$ which does not become infinite to an order as great as 1 for a finite value of z.

As in (3), by the application of the principle of partial fractions, we may suppose our function to be expressed as a sum in the form

$$H(z, u) = \sum_{\lambda} H_{\lambda}\left(\frac{1}{z - a_{\lambda}}, u\right) + H_0(z, u). \tag{24}$$

The function $H(z, u)$ will or will not be infinite for a point corresponding to the value $z = a_{\lambda}$ according as this is or is not the case for the corresponding element $H_{\lambda}(1/(z - a_{\lambda}), u)$ on the right. Writing the element in the form

$$H_{\lambda}\left(\frac{1}{z - a_{\lambda}}, u\right) = \frac{\sigma_1(u)}{z - a_{\lambda}} + \cdots + \frac{\sigma_{s-1}(u)}{(z - a_{\lambda})^{s-1}} + \frac{\sigma_s(u)}{(z - a_{\lambda})^s}, \tag{25}$$

where the numerators $\sigma(u)$ are polynomials in u of degree $n - 1$, we see that it will become infinite to the order s for some point corresponding to the value $z = a_{\lambda}$ unless we have

$$\sigma_s(u) = 0$$

for each of the values u corresponding to $z = a_{\lambda}$. The number of these values therefore cannot be greater than $n - 1$.

It follows then that in (24) the summation may be confined to values a_{λ} to which correspond branch or double points. Since for each of the $n - 1$ values of u corresponding to $z = a_{\lambda}$ we have $\sigma_s(u) = 0$, we may write

make use of the expression "infinite to the order $\frac{1}{2}$, 1, etc.," to denote an infinity of the character indicated by $(z - a)^{-\frac{1}{2}}$, $(z - a)^{-1}$, etc. The two forms of expression then differ in their signification when employed relatively to a branch point, but it is hoped that this word of explanation will prevent any confusion from arising in the mind of the reader.

$$\sigma_s(u) = \frac{hF(a_\lambda, u)}{u - b_\lambda},$$

where h is a constant.

In the case of a branch point (a_λ, b_λ) the function

$$\frac{\sigma_s(u)}{(z - a_\lambda)^s} = \frac{hF(a_\lambda, u)}{(z - a_\lambda)^s(u - b_\lambda)},$$

and therewith the element H_λ, here in question also, would evidently be infinite to the order $s - \frac{1}{2}$ so that we cannot have $s \geqq 2$. It follows therefore that $s = 1$.

In the case where a branch point corresponds to the value $z = a_\lambda$ then we shall have for our element:

$$H_\lambda\left(\frac{1}{z - a_\lambda}, u\right) = \frac{hF(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)}, \tag{26}$$

where h may have any arbitrary constant value. In the case where to $z = a_\lambda$ corresponds a double point let us suppose, if possible, that we have $s \geqq 2$. In this case the expression

$$\frac{\sigma_{s-1}(u)}{(z - a_\lambda)^{s-1}} + \frac{\sigma_s(u)}{(z - a_\lambda)^s}$$

must evidently not be infinite to an order greater than $s - 2$, for then it would be infinite to the order 1 at least, and this would hold also in regard to the element on the left of (25). The expression

$$\sigma_{s-1}(u) + \frac{\sigma_s(u)}{(z - a_\lambda)} = \sigma_{s-1}(u) + \frac{hF(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)}$$

must therefore vanish for each of the $n - 1$ points corresponding to the value $z = a_\lambda$ and in particular for each of the branches through the double point (a_λ, b_λ).

On indicating the directions of the tangents at the double point by m_1 and m_2 respectively, we must then have:

$$\sigma_{s-1}(b_\lambda) + \tfrac{1}{2}hm_1[F''_u(a_\lambda, u)]_{u=b_\lambda} = 0,$$

$$\sigma_{s-1}(b_\lambda) + \tfrac{1}{2}hm_2[F''_u(a_\lambda, u)]_{u=b_\lambda} = 0,$$

and therefore

$$h(m_1 - m_2)[F''_u(a_\lambda, u)]_{u=b_\lambda} = 0.$$

Since, however, $m_1 \neq m_2$ and $[F''_u(a_\lambda, u)]_{u=b_\lambda} \neq 0$, this is impossible unless we have $h = 0$ and consequently $\sigma_s(u) = 0$.

It follows therefore that we cannot have $s \geqq 2$.

For $s = 1$ we shall have

$$\frac{\sigma_1(u)}{z-a_\lambda}=\frac{hF(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)},$$

a function which is finite for all the $n-1$ points corresponding to $z=a_\lambda$.

In the case where a double point corresponds to the value $z=a_\lambda$, as in the case of a branch point, the element in (25) takes the form given in (26), and for our function $H(z, u)$ in (24) we may write

$$H(z, u)=\sum_e \frac{h_e F(a_e, u)}{(z-a_e)(u-b_e)}+H_0(z, u), \tag{27}$$

where the summation is supposed to extend to all those values a_e to which correspond branch or double points, where the coefficients h_e may have any arbitrary constant values, and where $H_0(z, u)$ is an arbitrary polynomial in z, u. This then represents the most general function $H(z, u)$ which does not become infinite to an order as great as 1 for any finite value of z.

The most general function of z, u which becomes infinite only at ∞ we should evidently derive from (27) on equating to 0 therein all those coefficients h_e corresponding to the branch points, or in other words, on supposing the summation in (27) to be extended only to those values a_e to which correspond double points.

Returning now to formula (23) we may by (27) express the same in the form:

$$\int R(z, u)dz=P(z, u)+\text{III}+\int\left\{\sum_e \frac{h_e F(a_e, u)}{(z-a_e)(u-b_e)}+H_0(z, u)\right\}dz. \tag{28}$$

§ 6.

We shall now occupy ourselves with the reduction of the integral

$$\int H_0(z, u)dz.$$

Supposing the degree of the polynomial $H_0(z, u)$ in z, u to be $n+r-1$ where $r \geqq 0$, we shall indicate this by the suffix $n+r-1$ and write

$$H_0(z, u)=T_{n+r-1}(z, u).$$

We may evidently express $T_{n+r-1}(z, u)$ in the form

$$T_{n+r-1}(z, u)=z^r t(z, u)+T'_{n+r-2}(z, u), \tag{29}$$

where the function $t(z, u)$ is a homogeneous polynomial in z, u of degree $n-1$ and where $T'_{n+r-2}(z, u)$ is a polynomial in z, u in general non-homogeneous and of degree $n+r-2$.

It is evident that the function $T_{n+r-1}(z, u)$ cannot be infinite to an order higher than $n+r-1$ for any of the n branches at ∞, and that it must be infinite to this order for one at least of these branches we may readily show.

Namely, on substituting in (29) for u in terms of z from the equation to a given one of the branches at ∞, the coefficient of z^{n+r-1} will be furnished by the element $z^r t(z, u)$ and will be the same as that obtained on substituting $u = \kappa z$ from the corresponding equation in (7). The term in z^{n+r-1} so obtained is

$$z^r t(z, \kappa z) = z^{n+r-1} t(1, \kappa),$$

and the coefficient $t(1, \kappa)$ of z^{n+r-1} will be different from 0 in the case of one at least of the n branches, as otherwise we should have

$$t(1, \kappa) = 0$$

for each of the n values $\kappa_1, \kappa_2, \cdots, \kappa_n$ and this equation would therefore be an identity since its degree in κ is $n - 1$. It would follow then also that we must have

$$t(z, u) = 0$$

identically, which however is not consistent with our hypothesis that our function $T_{n+r-1}(z, u)$ has the degree $n + r - 1$. We conclude then that a polynomial $T_{n+r-1}(z, u)$ becomes infinite to the order $n + r - 1$, that is, to an order equal to its degree, for one or more of the branches at ∞.

Now consider the integral

$$(30) \qquad \int z^r t(z, u) dz = \int \left\{ z^r t(z, u) - \frac{d}{dz} \frac{z^{r+1} t(z, u)}{n + r} \right\} dz + \frac{z^{r+1} t(z, u)}{n + r}.$$

The expression under the integral sign to the right of this identity evidently does not become infinite to the order $n + r - 1$ for any of the branches at ∞. It may be written in the form:

$$z^r t(z, u) - \frac{d}{dz} \frac{z^{r+1} t(z, u)}{n + r} = \frac{(n - 1) z^r t(z, u)}{n + r} - \frac{z^{r+1}}{n + r} \left(\frac{\partial t}{\partial z} + \frac{\partial t}{\partial u} \frac{du}{dz} \right)$$

$$= \frac{(n - 1) z^r t(z, u)}{n + r} - \frac{z^{r+1}}{n + r} \left(\frac{\partial t}{\partial z} + \frac{\partial t}{\partial u} \sum_e \frac{\phi_e(u)}{z - a_e} \right),$$

where for du/dz we have substituted its expression given in (5). From the last written form we immediately derive:

$$z^r t(z, u) - \frac{d}{dz} \frac{z^{r+1} t(z, u)}{n + r} = T''(z, u) + \sum_e \frac{\theta_e(u)}{z - a_e},$$

where $T''(z, u)$ is a polynomial in (z, u) of degree in u not greater than $n - 1$ and where the functions $\theta_e(u)$ are polynomials in u alone whose degrees are equal to or less than $n - 1$.

Further, the expression to the left of the formula just written, as we have seen above, does not become infinite to the order $n + r - 1$ for any of the branches at ∞ and as none of the elements in the summation

$$\sum_e \frac{\theta_e(u)}{z - a_e}$$

on the right becomes infinite to an order which is greater than $n - 2$, it follows that the polynomial $T''(z, u)$ also does not become infinite to the order $n + r - 1$ and that its degree in z, u therefore is not greater than $n + r - 2$. Indicating this fact by the suffix $n + r - 2$, we may write:

$$z^r t(z, u) - \frac{d}{dz}\frac{z^{r+1}t(z, u)}{n + r} = T''_{n+r-2}(z, u) + \sum_e \frac{\theta_e(u)}{z - a_e}; \tag{31}$$

and substituting the form so obtained for the function under the integral sign to the right of (30) we derive:

$$\int z^r t(z, u)dz = \int \left\{ T''_{n+r-2}(z, u) + \sum_e \frac{\theta_e(u)}{z - a_e} \right\} dz + \frac{z^{r+1}t(z, u)}{n + r}. \tag{32}$$

Combining this result with (29) we obtain:

$$\int T_{n+r-1}(z, u)dz = \int \left\{ T'_{n+r-2}(z, u) + T''_{n+r-2}(z, u) + \sum_e \frac{\theta_e(u)}{z - a_e} \right\} dz + \frac{z^{r+1}t(z, u)}{n + r},$$

and this we may more concisely write in the form

$$\int T_{n+r-1}(z, u)dz = g_{n+r}(z, u) + \sum_e \int \frac{\theta_e(u)}{z - a_e} dz + \int T_{n+r-2}(z, u)dz,$$

where

$$g_{n+r}(z, u) = \frac{z^{r+1}t(z, u)}{n + r}, \quad T_{n+r-2}(z, u) = T'_{n+r-2}(z, u) + T''_{n+r-2}(z, u).$$

Repeated application of the above process of reduction gives rise to a succession of integrals

$$\int T_{n+r-2}(z, u)dz, \quad \int T_{n+r-3}(z, u)dz, \cdots, \int T_{n-2}(z, u)dz,$$

and enables us to reduce our integral to an expression of the form

$$\int T_{n+r-1}(z, u)dz = G_{n+r}(z, u) + \sum_e \int \frac{\sigma_e(u)}{z - a_e} dz + \int T_{n-2}(z, u)dz, \tag{33}$$

where $G_{n+r}(z, u)$ and $T_{n-2}(z, u)$ are polynomials in z, u of degrees $n + r$ and $n - 2$ respectively, and where the functions $\sigma_e(u)$ are polynomials in u of degree $n - 1$.

From the manner in which the functions

$$\frac{\sigma_e(u)}{z-a_e}$$

arise in the process above it is further self-evident that they do not become infinite to an order as great as 1 for the values a_e, so that their sum will constitute a function of the type represented in (27) and may be expressed in the form:

$$\sum_e \frac{\sigma_e(u)}{z-a_e} = \sum_e \frac{\alpha_e F(a_e, u)}{(z-a_e)(u-b_e)},$$

where the coefficients α_e are constants and the summation is supposed to extend to those values a_e to which correspond branch or double points.

For formula (33) we may therefore write:

$$(34)\quad \int T_{n+r-1}(z, u)dz = G_{n+r}(z, u) + \sum_e \int \frac{\alpha_e F(a_e, u)}{(z-a_e)(u-b_e)}dz + \int T_{n-2}(z, u)dz;$$

and on substituting for $\int H_0(z, u)dz = \int T_{n+r-1}(z, u)dz$ in (28) the expression here obtained, we arrive at a reduced form for our original integral $\int R(z, u)dz$ which may be written.

$$(35)\quad \int R(z, u)dz = \bar{G}(z, u) + \text{III} + \int \left\{ \sum_e \frac{\beta_e F(a_e, u)}{(z-a_e)(u-b_e)} + T_{n-2}(z, u) \right\} dz,$$

where III is an integral of the third kind, $\bar{G}(z, u)$ a rational function of z, u, $T_{n-2}(z, u)$ a polynomial in z, u of degree $n-2$ and the β_e's constants, the summation with regard to e being extended to all the branch and double points of our curve.

§ 7.

Our reductions of the integral $\int R(z, u)dz$ in the above have so far consisted in getting rid of various infinities of the function under the sign of integration and accounting for the same by a rational function $\bar{G}(z, u)$, its logarithmic infinities at the same time being separated from the main integral and incorporated in an integral of the third kind III.

We shall now consider the remaining integral to the right of (35) with a view to a further reduction. The expression under the integral sign is of the type represented in (27), which for no finite value of z becomes infinite to an order which is as great as 1. The integral itself,

$$(36)\qquad \int \left\{ \sum_e \frac{\beta_e F(a_e, u)}{(z-a_e)(u-b_e)} + T_{n-2}(z, u) \right\} dz,$$

will therefore be finite for all finite values of z and for $z = \infty$ will not become

infinite to an order which is greater than $n-1$ since the function under the sign of integration is evidently not infinite to an order which is greater than $n-2$ for any of the points at ∞. If now we would attempt a further elimination of infinities from under the sign of integration by the aid of a rational function of z, u and that without introducing any new infinities, the most general function at our disposal is included under the type (27) and will evidently have the form

$$\sum_e \frac{\gamma_e F(a_e, u)}{(z-a_e)(u-b_e)} + T_{n-1}(z, u), \tag{37}$$

where $T_{n-1}(z, u)$ is a polynomial in z, u of degree $n-1$, the summation with regard to e being supposed to extend to all the double points and the coefficients γ_e having any arbitrary values; for this is the most general rational function whose infinities are all at ∞ and which does not there become infinite to an order which is greater than $n-1$.

The number of coefficients which the polynomial $T_{n-1}(z, u)$ places at our disposal is $\frac{1}{2}n(n+1)$, or we may say $\frac{1}{2}n(n+1)-1$ on rejecting the constant term which could only serve to modify the arbitrary constant of integration in (36), and the number of the coefficients γ_e in the expression (37) is d, the number of the double points of our curve. In all, then, the number of undetermined coefficients which the expression (37) places at our disposal is

$$d + \tfrac{1}{2}n(n+1) - 1.$$

Now the integral in (36) will evidently in general become infinite to the order $n-1$ for each of the n branches at ∞, possessing therefore in all $n(n-1)$ infinities all situated at ∞. If then each of these infinities requires an extra available constant in order to eliminate it from under the sign of integration, the number of available constants placed at our disposal by the expressions (37) will fall short of that requisite by at least

$$n(n-1) - d - \tfrac{1}{2}n(n+1) + 1,$$

that is, by at least

$$p = \tfrac{1}{2}(n-1)(n-2) - d.$$

In order to make good this deficit I shall introduce p more constants, introducing however with each one of them at the same time a new infinity. Namely instead of (37) I shall consider the expression

$$\sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} + T_{n-1}(z, u), \tag{38}$$

in which the summation is supposed to extend not only to the double points, but also to p additional points (a_λ, b_λ) which do not coincide with double points but may otherwise be regarded as arbitrarily chosen. For these p points,

in case the corresponding coefficients γ are not zero, the function in (38) will evidently become infinite to the first order. For all other finite points it will be finite and for the n points at ∞ it will in general become infinite to the order $n-1$.

In the sequel we shall find that the $n(n-1)$ constants placed at our disposal by the expression (38) are all available and none of them superfluous for the purpose of ridding ourselves of all the infinities at ∞ of the general integral of the type (36).

§ 8.

Consider the expression

$$(39)\quad \int \left\{ \sum_{\epsilon} \frac{\beta_{\epsilon} F(a_{\epsilon}, u)}{(z-a_{\epsilon})(u-b_{\epsilon})} + T_{n-2}(z, u) \right\} dz - \left\{ \sum_{\lambda=1}^{d+p} \frac{\gamma_{\lambda} F(a_{\lambda}, u)}{(z-a_{\lambda})(u-b_{\lambda})} + T_{n-1}(z, u) \right\},$$

obtained on subtracting the function in (38) from the integral in (36). Regarding the function under the sign of integration as given and the coefficients in the subtracted function as undetermined, we shall attempt to determine the latter so that the expression in (39) may have no infinities at ∞.

From the equation to our curve in (1) we have:

$$\frac{F(a_{\lambda}, u)}{(z-a_{\lambda})(u-b_{\lambda})} = \frac{F(a_{\lambda}, u) - F(a_{\lambda}, b_{\lambda})}{(z-a_{\lambda})(u-b_{\lambda})} = \frac{\sum e_{r,s} a_{\lambda}^{r} u^{s} - \sum e_{r,s} a_{\lambda}^{r} b_{\lambda}^{s}}{(z-a_{\lambda})(u-b_{\lambda})}$$

$$= \frac{\sum e_{r,s} a_{\lambda}^{r} (u^{s} - b_{\lambda}^{s})}{(z-a_{\lambda})(u-b_{\lambda})},$$

whence

$$(40)\quad \frac{F(a_{\lambda}, u)}{(z-a_{\lambda})(u-b_{\lambda})} = (z-a_{\lambda})^{-1} \sum e_{r,s} a_{\lambda}^{r} (u^{s-1} + u^{s-2} b_{\lambda} + \cdots + b_{\lambda}^{s-1})$$

$$= (z-a_{\lambda})^{-1} \sum_{s=1}^{n} \{a_{\lambda}, b_{\lambda}\}_{s} u^{n-s},$$

where

$$\{a_{\lambda}, b_{\lambda}\}_{s} = \sum_{\sigma=0}^{s-1} \sum_{\rho=0}^{\sigma} e_{\rho, n-\sigma} a_{\lambda}^{\rho} b_{\lambda}^{s-\sigma-1}.$$

We may then write:

$$\frac{F(a_{\lambda}, u)}{(z-a_{\lambda})(u-b_{\lambda})} = \left\{ \frac{1}{z} + \frac{a_{\lambda}}{z^{2}} + \cdots + \frac{a_{\lambda}^{n-2}}{z^{n-1}} + \frac{a_{\lambda}^{n-1}}{z^{n-1}(z-a_{\lambda})} \right\} \sum_{s=1}^{n} \{a_{\lambda}, b_{\lambda}\}_{s} u^{n-s}$$

$$= \left\{ \frac{1}{z} + \frac{a_{\lambda}}{z^{2}} + \cdots + \frac{a_{\lambda}^{n-2}}{z^{n-1}} \right\} \sum_{s=1}^{n} \{a_{\lambda}, b_{\lambda}\}_{s} u^{n-s} + P_{\lambda}(z, u),$$

where $P_\lambda(z, u)$ is a rational function of z, u which is equal to 0 for $z = \infty$. This we may evidently further reduce to the form:

$$\frac{F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} = \sum_{s=1}^{n-1} \sum_{r=1}^{n-s} a_\lambda^{r-1} \{a_\lambda, b_\lambda\}_s z^{-r} u^{n-s} + \bar{P}_\lambda(z, u), \tag{41}$$

where $\bar{P}_\lambda(z, u)$ is a rational function of z, u which is equal to 0 for $z = \infty$. The second of the two summations in (39) may then be written in the form:

$$\begin{aligned} \sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} &= \sum_{\lambda=1}^{d+p} \sum_{s=1}^{n-1} \sum_{r=1}^{n-s} \gamma_\lambda a_\lambda^{r-1} \{a_\lambda, b_\lambda\}_s z^{-r} u^{n-s} + \bar{P}(z, u) \\ &= \sum_{s=1}^{n-1} \sum_{r=1}^{n-s} c_{n-s-r,\, n-s} z^{-r} u^{n-s} + \bar{P}(z, u) \\ &= \sum_{s=1}^{n-2} \sum_{r=1}^{n-s-1} c_{n-s-r,\, n-s} z^{-r} u^{n-s} + P(z, u), \end{aligned} \tag{42}$$

where $\bar{P}(z, u)$ and $P(z, u)$ are rational functions of z, u which are finite for $z = \infty$, the former function in particular having the value 0 for this value of the variable, and where for the coefficients c we have

$$c_{n-s-r,\, n-s} = \sum_{\lambda=1}^{d+p} \gamma_\lambda a_\lambda^{r-1} \{a_\lambda, b_\lambda\}_s.$$

For the suffixes of the coefficients c we shall find it convenient to choose, as we have done here, the sum of the exponents of z and u and the exponent of u respectively.

On writing

$$\gamma_{i\kappa} = \sum_{\lambda=1}^{d+p} \gamma_\lambda a_\lambda^i b_\lambda^\kappa, \tag{43}$$

we shall evidently have

$$c_{n-s-r,\, n-s} = \sum_{\sigma=0}^{s-1} \sum_{\rho=0}^{\sigma} e_{\rho,\, n-\sigma} \gamma_{r-1+\rho,\, s-1-\sigma}. \tag{44}$$

Since the exponents of z which appear in the summation to the right of (42) are all negative whereas in the polynomial $T_{n-1}(z, u)$ in (39) no such exponents make their appearance, it will not be inconsistent with the notation employed above if we write:

$$T_{n-1}(z, u) = \sum_{s=1}^{n} \sum_{r=0}^{s-1} c_{n-s+r,\, n-s} z^r u^{n-s}. \tag{45}$$

Combining (42) and (45) we have:

$$T_{n-1}(z,u)+\sum_{\lambda=1}^{d+p}\frac{\gamma_\lambda F(a_\lambda,u)}{(z-a_\lambda)(u-b_\lambda)}=\sum_{s=1}^{n}\sum_{r=0}^{s-1}c_{n-s+r,\,n-s}z^r u^{n-s}$$

$$+\sum_{s=1}^{n-1}\sum_{r=1}^{n-s}c_{n-s-r,\,n-s}z^{-r}u^{n-s}+\bar{P}(z,u) \tag{46}$$

$$=\sum_{s=1}^{n}\sum_{r=-(n-s)}^{s-1}c_{n-s+r,\,n-s}z^r u^{n-s}+\bar{P}(z,u).$$

This again we may write in the form:

$$\sum_{\lambda=1}^{d+p}\frac{\gamma_\lambda F(a_\lambda,u)}{(z-a_\lambda)(u-b_\lambda)}+T_{n-1}(z,u)=\sum_{\nu=0}^{n-1}z^{-\nu}p_\nu(z,u)+\bar{P}(z,u) \tag{47}$$

$$=\sum_{\nu=0}^{n-2}z^{-\nu}p_\nu(z,u)+P(z,u),$$

where

$$p_\nu(z,u)=\sum_{s=1}^{n}c_{n-1-\nu,\,n-s}z^{s-1}u^{n-s},$$

is a homogeneous polynomial in (z,u) of degree $n-1$.

The expression under the sign of integration in (39) we can evidently also reduce to a form like that on the right of (47). In this form further we may here omit terms of degree $n-1$ since in the expression under the sign of integration terms of order higher than $n-2$ do not present themselves.

We may then write

$$\sum_{e}\frac{\beta_e F(a_e,u)}{(z-a_e)(u-b_e)}+T_{n-2}(z,u)=z^{-1}q_1(z,u)+z^{-2}q_2(z,u)$$

$$+\cdots+z^{-(n-1)}q_{n-1}(z,u)+Q(z,u), \tag{48}$$

where $q_1(z,u),\ \cdots,\ q_{n-1}(z,u)$ are homogeneous polynomials in $z,\ u$ of degree $n-1$ and where $Q(z,u)$ is a rational function of $z,\ u$ which becomes 0 for $z=\infty$.

From (47) and (48) the expression (39) may be written in the form:

$$\int\{z^{-1}q_1(z,u)+z^{-2}q_2(z,u)+\cdots+z^{-(n-1)}q_{n-1}(z,u)+Q(z,u)\}dz$$

$$-\{p_0(z,u)+z^{-1}p_1(z,u)+\cdots+z^{-(n-2)}p_{n-2}(z,u)+P(z,u)\}. \tag{49}$$

Supposing now that the expression under the sign of integration is given while the subtracted expression is as yet undetermined, we shall attempt to determine the coefficients c of the polynomials $p_0(z,u),\ p_1(z,u),\ \cdots,\ p_{n-2}(z,u)$ in such way that the expression in (49) may be finite at ∞.

§ 9.

We have seen in § 6 that a function $z^r t(z, u)$, where $t(z, u)$ is a homogeneous polynomial in z, u of degree $n-1$, becomes infinite to the order $n+r-1$ but to no higher order, for one or more of the branches at ∞. We had there assumed $r \geq 0$ but the same evidently holds also for negative values of r.

The only portion of the expression (49) which might possibly involve infinities at ∞ of order $n-1$ is then evidently

$$\int z^{-1} q_1(z, u) dz - p_0(z, u). \tag{50}$$

We can however so choose $p_0(z, u)$ that the expression just written does not become infinite to an order higher than $n-2$. For, on taking

$$p_0(z, u) = \frac{q_1(z, u)}{n-1}$$

we shall have:

$$\int z^{-1} q_1(z, u) dz - p_0(z, u) = \int \left\{ z^{-1} q_1(z, u) - \frac{1}{n-1} \frac{d}{dz} q_1(z, u) \right\} dz \tag{51}$$

and the expression under the integral sign to the right evidently does not become infinite to the order $n-2$ for any of the branches at ∞. On substituting for du/dz its expression given in (5) we obtain the function

$$\frac{1}{n-1} \frac{d}{dz} q_1(z, u) = \frac{1}{n-1} \left\{ \frac{\partial q_1}{\partial z} + \frac{\partial q_1}{\partial u} \frac{du}{dz} \right\}$$

as a rational function of z, u which does not become infinite at ∞ to an order higher than $n-2$ nor elsewhere to an order as high as 1. It is then a function of the type represented in (48) and may therefore be similarly represented. The total expression under the sign of integration to the right of (51) may then also be represented in that form which appears to the right of (48) and we may consequently write:

$$z^{-1} q_1(z, u) - \frac{1}{n-1} \frac{d}{dz} q_1(z, u) = z^{-1} q_1'(z, u) + z^{-2} q_2'(z, u) + \cdots + z^{-(n-1)} q_{n-1}'(z, u) + Q'(z, u),$$

where $q_1'(z, u), \cdots, q_{n-1}'(z, u)$ are homogeneous polynomials in z, u of degree $n-1$ and where $Q'(z, u)$ is a rational function of z, u which becomes 0 for $z = \infty$. Further, since the expression here represented does not become infinite to the order $n-2$ for any of the branches at ∞, we must have identically

$$q_1'(z, u) = 0.$$

From (51) we shall then have:

$$(52)\quad \int z^{-1}q_1(z,u)dz - p_0(z,u) = \int \{z^{-2}q_2'(z,u) + \cdots + z^{-(n-1)}q_{n-1}'(z,u) + Q'(z,u)\}dz,$$

and with the aid of this formula we may write our expression in (49) in the form:

$$(53)\quad \int \{z^{-2}q_2''(z,u) + \cdots + z^{-(n-1)}q_{n-1}''(z,u) + Q''(z,u)\}dz - \{z^{-1}p_1(z,u) + z^{-2}p_2(z,u) + \cdots + z^{-(n-2)}p_{n-2}(z,u) + P(z,u)\},$$

where

$$q_2''(z,u) = q_2(z,u) + q_2'(z,u),$$
$$\cdots\cdots\cdots$$
$$q_{n-1}''(z,u) = q_{n-1}(z,u) + q_{n-1}'(z,u);$$
$$Q''(z,u) = Q(z,u) + Q'(z,u).$$

The expression (53) evidently will not become infinite to an order greater than $n-2$ for any of the branches at ∞ and we may further so choose $p_1(z,u)$ that it will not become infinite even to this order.

For the only portion of the expression (53) which might possibly involve infinities at ∞ of order $n-2$ is evidently

$$\int z^{-2}q_2''(z,u)dz - z^{-1}p_1(z,u),$$

and on choosing

$$p_1(z,u) = \frac{q_2''(z,u)}{n-2},$$

we shall have:

$$(54)\quad \int z^{-2}q_2''(z,u)dz - z^{-1}p_1(z,u) = \int \left\{z^{-2}q_2''(z,u) - \frac{d}{dz}\frac{z^{-1}q_2''(z,u)}{n-2}\right\}dz,$$

where, on substituting for u in terms of z from the equation to a branch at ∞, the expression under the sign of integration on the right will evidently not be infinite to an order higher than $n-4$ and the integral itself therefore to an order not higher than $n-3$.

This expression we may write in the form:

$$z^{-2}q_2''(z,u) - \frac{1}{n-2}\frac{d}{dz}z^{-1}q_2''(z,u) = \frac{n-1}{n-2}z^{-2}q_2''(z,u) - \frac{z^{-1}}{n-2}\frac{dq_2''(z,u)}{dz}.$$

As in the case of $dq_1(z,u)/dz$ so here $dq_2''(z,u)/dz$—and in fact the derivative of any homogeneous polynomial in z, u of degree $n-1$—is of the type of function represented in (48), and the expression just written may therefore evidently be written in the form:

$$\frac{n-1}{n-2}z^{-2}q_2''(z,u)-\frac{z^{-1}}{n-2}\frac{dq_2''(z,u)}{dz}=z^{-2}q_2'''(z,u)+z^{-3}q_3'''(z,u)\cdots + z^{-(n-1)}q_{n-1}'''(z,u)+Q'''(z,u),$$

where $q_2'''(z,u),\cdots,q_{n-1}'''(z,u)$ are homogeneous polynomials in z, u of degree $n-1$ and where $Q'''(z,u)$ is a rational function of z, u which becomes 0 for $z=\infty$. Further, since the expression here represented does not become infinite to the order $n-3$ for any of the branches at ∞, we must have identically

$$q_2'''(z,u)=0.$$

From (54) we shall then have:

$$(55)\quad \int z^{-2}q_2''(z,u)dz-z^{-1}p_1(z,u)=\int\{z^{-3}q_3'''(z,u)+z^{-4}q_4'''(z,u)+\cdots + z^{-(n-1)}q_{n-1}'''(z,u)+Q'''(z,u)\}dz,$$

and with the aid of this formula we may write the expression in (53) and therefore that in (49) in the form

$$(56)\quad \int\{z^{-3}q_3^{(4)}(z,u)+\cdots+z^{-(n-1)}q_{n-1}^{(4)}(z,u)+Q^{(4)}(z,u)\}dz-\{z^{-2}p_2(z,u)+\cdots+z^{-(n-2)}p_{n-2}(z,u)+P(z,u)\},$$

where

$$q_3^{(4)}(z,u)=q_3''(z,u)+q_3'''(z,u),$$

. . . . . .

$$q_{n-1}^{(4)}(z,u)=q_{n-1}''(z,u)+q_{n-1}'''(z,u);$$

$$Q^{(4)}(z,u)=Q''(z,u)+Q'''(z,u).$$

The expression (56) will evidently not become infinite to an order greater than $n-3$ for any of the branches at ∞ and we might so choose $p_2(z,u)$ that it would not become infinite even to this order. Suppose now that by proper choice of the polynomials $p_2(z,u),\cdots,p_{r-1}(z,u)$ we have reduced (56) and therewith (49) to the form

$$(57)\quad \int\{z^{-(r+1)}q_{r+1}^{(2r)}(z,u)+\cdots+z^{-(n-1)}q_{n-1}^{(2r)}(z,u)+Q^{(2r)}(z,u)\}dz-\{z^{-r}p_r(z,u)+\cdots+z^{-(n-2)}p_{n-2}(z,u)\}+P(z,u),$$

where $q_{r+1}^{(2r)}(z,u),\cdots,q_{n-1}^{(2r)}(z,u)$ are homogeneous polynomials in z, u of degree $n-1$ and where $Q^{(2r)}(z,u)$ is a rational function of z, u which becomes 0 for $z=\infty$. This expression will evidently not become infinite to an order greater than $n-r-1$ for any of the branches at ∞ and we may so choose $p_r(z,u)$ that it will not become infinite to an order greater than $n-r-2$.

For the only portion of the expression (57) which might possibly involve infinities at ∞ of order $n-r-1$ is evidently

$$\int z^{-(r+1)} q_{r+1}^{(2r)}(z, u)\,dz - z^{-r} p_r(z, u),$$

and on choosing

$$p_r(z, u) = \frac{q_{r+1}^{(2r)}(z, u)}{n-r-1}$$

we shall have:

$$\int z^{-(r+1)} q_{r+1}^{(2r)}(z, u)\,dz - z^{-r} p_r(z, u) = \int \left\{ z^{-(r+1)} q_{r+1}^{(2r)}(z, u) - \frac{d}{dz} \frac{z^{-r} q_{r+1}^{(2r)}(z, u)}{n-r-1} \right\} dz, \tag{58}$$

where, on substituting for u in terms of z from the equation to a branch at ∞, the expression under the sign of integration on the right will evidently not be infinite to an order higher than $n-r-3$ and the integral itself therefore to an order not higher than $n-r-2$.

This expression we may write in the form:

$$z^{-(r+1)} q_{r+1}^{(2r)}(z, u) - \frac{1}{n-r-1} \frac{d}{dz} z^{-r} q_{r+1}^{(2r)}(z, u) = \frac{n-1}{n-r-1} z^{-(r+1)} q_{r+1}^{(2r)}(z, u) - \frac{z^{-r}}{n-r-1} \frac{dq_{r+1}^{(2r)}(z, u)}{dz}.$$

The derivative $dq_{r+1}^{(2r)}(z, u)/(dz)$ is of the type of function represented in (48) and the expression with which we here have to do may evidently therefore be represented in the form:

$$\frac{n-1}{n-r-1} z^{-(r+1)} q_{r+1}^{(2r)}(z, u) - \frac{z^{-r}}{n-r-1} \frac{dq_{r+1}^{(2r)}(z, u)}{dz} = z^{-(r+1)} q_{r+1}^{(2r+1)}(z, u) + z^{-(r+2)} q_{r+2}^{(2r+1)}(z, u) + \cdots + z^{-(n-1)} q_{n-1}^{(2r+1)}(z, u) + Q^{(2r+1)}(z, u),$$

where $q_{r+1}^{(2r+1)}(z, u), \cdots, q_{n-1}^{(2r+1)}(z, u)$ are homogeneous polynomials in z, u of degree $n-1$ and where $Q^{(2r+1)}(z, u)$ is a rational function of z, u which becomes 0 for $z = \infty$. Further since, as we have just seen, the expression here represented does not become infinite to an order higher than $n-r-3$ for any of the branches at ∞, we must have identically

$$q_{r+1}^{(2r+1)}(z, u) = 0.$$

From (58) we shall then have:

$$\int z^{-(r+1)} q_{r+1}^{(2r)}(z, u)\,dz - z^{-r} p_r(z, u) = \int \{ z^{-(r+2)} q_{r+2}^{(2r+1)}(z, u) + \cdots + z^{-(n-1)} q_{n-1}^{(2r+1)}(z, u) + Q^{(2r+1)}(z, u) \}\, dz, \tag{59}$$

and with the aid of this formula we may write the expression in (57) and therewith that in (49) in the form

$$\int \{z^{-(r+2)}q_{r+2}^{(2r+2)}(z, u) + \cdots + z^{-(n-1)}q_{n-1}^{(2r+2)}(z, u) + Q^{(2r+2)}(z, u)\}dz$$
$$- \{z^{-(r+1)}p_{r+1}(z, u) + \cdots + z^{-(n-2)}p_{n-2}(z, u) + P(z, u)\} \tag{60}$$

where

$$q_{r+2}^{(2r+2)}(z, u) = q_{r+2}^{(2r)}(z, u) + q_{r+2}^{(2r+1)}(z, u),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$q_{n-1}^{(2r+2)}(z, u) = q_{n-1}^{(2r)}(z, u) + q_{n-1}^{(2r+1)}(z, u);$$
$$Q^{(2r+2)}(z, u) = Q^{(2r)}(z, u) + Q^{(2r+1)}(z, u).$$

By induction then we see that a proper choice of the homogeneous polynomials $p_0(z, u)$, $p_1(z, u)$, $\cdots$, $p_{n-2}(z, u)$ will enable us to reduce (49) to the form

$$\int Q^{(2n-2)}(z, u)dz - P(z, u), \tag{61}$$

where $Q^{(2n-2)}(z, u)$ is a rational function of (z, u) which becomes equal to 0 for $z = \infty$. Since also $P(z, u)$ is finite for $z = \infty$ it follows that the expression (61) cannot become algebraically infinite at ∞. That it cannot become logarithmically infinite is also evident, for the integral in the equivalent expression (49), being only another form for the integral in (39), does not become infinite logarithmically.

We see then that the $n(n-1)$ coefficients in the $n-1$ polynomials $p_0(z, u)$, $p_1(z, u)$, $\cdots$, $p_{n-2}(z, u)$ may be so chosen that the expression in (49) does not become infinite for $z = \infty$, and further we may show that there is only one way of so choosing them. For if we had two systems of functions $p_0(z, u)$, $p_1(z, u)$, $\cdots$, $p_{n-2}(z, u)$, $P(z, u)$ and $p_0'(z, u)$, $p_1'(z, u)$, $\cdots$, $p_{n-2}'(z, u)$, $P'(z, u)$ corresponding to two determinations of these coefficients, the expression

$$\int \{z^{-1}q_1(z, u) + z^{-2}q_2(z, u) + \cdots + z^{-(n-1)}q_{n-1}(z, u) + Q(z, u)\}dz$$
$$- \{p_0'(z, u) + z^{-1}p_1'(z, u) + \cdots + z^{-(n-2)}p_{n-2}'(z, u) + P'(z, u)\},$$

and that in (49) would both be finite for $z = \infty$. The difference between these two expressions would then be finite for $z = \infty$ and, since the functions $P(z, u)$ and $P'(z, u)$ are finite for this value of z, this also would be the case for the difference,

$$\{p_0(z, u) + z^{-1}p_1(z, u) + \cdots + z^{-(n-2)}p_{n-2}(z, u)\}$$
$$- \{p_0'(z, u) + z^{-1}p_1'(z, u) + \cdots + z^{-(n-2)}p_{n-2}'(z, u)\}.$$

From this however evidently follows that we must have identically:

$$p_0(z, u) = p_0'(z, u), \quad p_1(z, u) = p_1'(z, u), \quad \cdots, \quad p_{n-2}(z, u) = p_{n-2}'(z, u).$$

We see then that there is one and only one determination of the coefficients in the $n-1$ polynomials $p_0(z, u), p_1(z, u), \cdots, p_{n-2}(z, u)$, such that the expression in (49) does not become infinite for $z = \infty$.

§ 10.

The $n(n-1)$ coefficients of the polynomials $p_0(z, u), \cdots, p_{n-2}(z, u)$ in (47) are those which have been indicated by

$$\begin{array}{llll} c_{n-1,n-1}, & c_{n-1,n-2}, & \cdots, & c_{n-1,0}, \\ c_{n-2,n-1}, & c_{n-2,n-2}, & \cdots, & c_{n-2,0}, \\ c_{n-3,n-1}, & c_{n-3,n-2}, & \cdots, & c_{n-3,0}, \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \\ c_{1,n-1}, & c_{1,n-2}, & \cdots, & c_{1,0}. \end{array}$$

To these coefficients we shall suppose to be assigned that system of values for which the expression (49) does not become infinite at ∞.

With those coefficients in which the first suffix is equal to or greater than the second, we shall construct a polynomial

$$T_{n-1}(z, u) = \sum_{s=1}^{n} \sum_{r=0}^{s-1} c_{n-s+r,\, n-s} z^r u^{n-s}, \tag{62}$$

in which the constant term c_{00} may have any arbitrary value.

Corresponding to the $\frac{1}{2}(n-1)(n-2)$ coefficients

$$\begin{array}{l} c_{n-2,n-1}; \\ c_{n-3,n-1}, c_{n-3,n-2}; \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ c_{1,n-1}, c_{1,n-2}, \cdots, c_{1,2}, \end{array} \tag{63}$$

in which the first suffix is smaller than the second, we shall have as many equations of the type given in (44). They will be obtained on giving to r and s respectively in (44) the values

$$r = 1, 2, \cdots, n-s-1; \quad s = 1, 2, \cdots, n-2. \tag{64}$$

This system of $\frac{1}{2}(n-1)(n-2)$ equations we shall attempt to satisfy by a proper choice of the quantities $\gamma_{i\kappa}$ on the right. The number of these quantities which present themselves in the $\frac{1}{2}(n-1)(n-2)$ equations is just equal to the number of the equations as we may readily show. For the greatest value for the sum of the suffixes i and κ in a term on the right of (44) is equal to $r+s-2$, and $r+s-2$, as we see from (64), cannot exceed $n-3$ in value. It follows therefore that the sum of the suffixes in the quantities $\gamma_{i\kappa}$ which ap-

pear in our equations, is in no case greater than $n-3$. Now the number of possible pairs of integers i, κ whose sum is equal to or less that $n-3$ is $\frac{1}{2}(n-1)(n-2)$, and that all the $\frac{1}{2}(n-1)(n-2)$ corresponding quantities $\gamma_{i\kappa}$ actually present themselves in our system of equations is immediately apparent. In fact each one of them appears in some one of the equations (44) with $e_{0,n}$ as coefficient as we see on taking $\rho=0$, $\sigma=0$ and substituting for r, s in the corresponding term $e_{0,n}\gamma_{r-1,s-1}$ the several pairs of values furnished by (64).

The number of the undetermined quantities $\gamma_{i\kappa}$ which present themselves in the system of equations (44) is then just equal to $\frac{1}{2}(n-1)(n-2)$, the number of these equations.

From the equations (44) we shall now seek to determine the quantities $\gamma_{i\kappa}$ in terms of the quantities $c_{n-s-r,n-s}$. For this purpose we shall write our equations in the form:

$$(65)\quad c_{n-s-r,n-s}=e_{0,n}\gamma_{r-1,s-1}+e_{1,n-1}\gamma_{r,s-2}+e_{2,n-2}\gamma_{r+1,s-3}+\cdots+e_{s-2,n-s+2}\gamma_{r+s-3,1}+e_{s-1,n-s+1}\gamma_{r+s-2,0}+\Gamma_{r-1,s-1},$$

where $\Gamma_{r-1,s-1}$ depends only on quantities $\gamma_{i\kappa}$, the sum of whose suffixes is less than $r+s-2$.

The number of equations of the type (65), in which the sum of the suffixes of the quantities γ in evidence on the right has a given value $r+s+2=\sigma$, is $\sigma+1$ and these $\sigma+1$ equations may evidently be written in the form:

$$\begin{aligned}
e_{0,n}\gamma_{0,\sigma}+e_{1,n-1}\gamma_{1,\sigma-1}+e_{2,n-2}\gamma_{2,\sigma-2}+\cdots+e_{\sigma-1,n-\sigma+1}\gamma_{\sigma-1,1}+e_{\sigma,n-\sigma}\gamma_{\sigma,0}&=c_{n-\sigma-2,n-\sigma-1}-\Gamma_{0,\sigma},\\
e_{0,n}\gamma_{1,\sigma-1}+e_{1,n-1}\gamma_{2,\sigma-2}+\cdots+e_{\sigma-2,n-\sigma+2}\gamma_{\sigma-1,1}+e_{\sigma-1,n-\sigma+1}\gamma_{\sigma,0}&=c_{n-\sigma-2,n-\sigma}-\Gamma_{1,\sigma-1},\\
e_{0,n}\gamma_{2,\sigma-2}+\cdots+e_{\sigma-3,n-\sigma+3}\gamma_{\sigma-1,1}+e_{\sigma-2,n-\sigma+2}\gamma_{\sigma,0}&=c_{n-\sigma-2,n-\sigma+1}-\Gamma_{2,\sigma-2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
e_{0,n}\gamma_{\sigma-1,1}+e_{1,n-1}\gamma_{\sigma,0}&=c_{n-\sigma-2,n-2}-\Gamma_{\sigma-1,1},\\
e_{0,n}\gamma_{\sigma,0}&=c_{n-\sigma-2,n-1}-\Gamma_{\sigma,0}.
\end{aligned}\qquad(66)$$

If now we suppose that the quantities $\gamma_{i\kappa}$ in which the sum of the suffixes is less than σ—and by consequence also the quantities Γ to the right of (66)—have been obtained in terms of the quantities c, the system of equations (66) will

determine in terms of these latter quantities the quantities $\gamma_{i\kappa}$ also, in which the sum of the suffixes is equal to σ. For the determinant of the coefficients on the left of (66) is different from 0 and has in fact the value $e_{0,n}^{\sigma+1} = 1$, since in our original equation $F(z, u) = 0$ we assumed $e_{0,n} = 1$.

More precisely still, on assuming that the quantities $\gamma_{i\kappa}$, in which the sum of the suffixes is less than σ, are linearly expressible in terms of the quantities c, it follows from the system of equations (66) that the quantities $\gamma_{i\kappa}$ in which the sum of the suffixes is equal to σ are also so expressible, and these equations furnish us with their expressions in terms of the c's.

Consider now the case $\sigma = 0$. In this case the system of equations (66) reduces to the single equation

$$e_{0,n}\gamma_{0,0} = c_{n-2,n-1},$$

so that the quantity $\gamma_{0\,0}$, in accord with our hypothesis above, is linearly expressible in terms of the quantities c and in fact is equal to one of them, namely $c_{n-2,n-1}$. It follows therefrom by our inductive reasoning above, that the quantities $\gamma_{0\,1}$ and $\gamma_{1\,0}$ corresponding to the case $\sigma = 1$ are also linearly expressible in terms of the quantities c, and by successive induction we see that the quantities $\gamma_{i\kappa}$ for which $\sigma(= i + \kappa)$ has one of the values $2, 3, \cdots, n-3$, are all linearly expressible in terms of the quantities c, the expressions for the same being furnished by the system of equations (44) supposed for that purpose to be grouped in a series of sets of the type given in (66).

The values of the $\frac{1}{2}(n-1)(n-2)$ quantities $\gamma_{i\kappa}$, so obtained, we may suppose to be substituted on the left in the corresponding equations of the type (43), when expressions for the $d + p = \frac{1}{2}(n-1)(n-2)$ quantities γ_λ in terms of our quantities $\gamma_{i\kappa}$ will be furnished by these equations in case we have

$$|a_\lambda^i b_\lambda^\kappa| \neq 0 \qquad (i+\kappa=0, 1, 2, \cdots, n-3), \tag{67}$$

where by $|a_\lambda^i b_\lambda^\kappa|$ we indicate the determinant of the coefficients of the quantities γ_λ to the right of our equations (43). That the determination so made of the quantities γ_λ is unique is self-evident.

In the case where we have $|a_\lambda^i b_\lambda^\kappa| = 0$ it will be impossible to satisfy the system of equations (43) by any set of values γ_λ unless it happens that a certain relation exists between the quantities $\gamma_{i\kappa}$. Such a relation however does not in general exist, as we shall show later on.

Assuming then that our system of points (a_λ, b_λ) has been so chosen that the determinant above has a value other than 0, or, what evidently amounts to the same thing, in such way that the points do not all lie on a curve of degree $n + 3$, we shall suppose that the function

$$\sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + T_{n-1}(z, u)$$

in (39) has been constructed with the values of the $d + p$ quantities γ_λ furnished by the equations (43) and with the polynomial $T_{n-1}(z, u)$ obtained in (62). Apart from the arbitrary constant in $T_{n-1}(z, u)$ the expression (39) so determined is unique.

We have shown that if the expression (39) is to be finite for all the branches at ∞ it must be determined as just stated. Conversely in case it has the determination here in question it will be finite for $z = \infty$.

For the coefficients γ_λ in (39) have been derived through the equations (43) and (44) from certain of the coefficients c in the polynomials $p_0(z, u)$, $p_1(z, u)$, $\cdots$, $p_{n-2}(z, u)$, these polynomials having been at the same time assumed by us to be those for which the expression (49) is finite at ∞; and conversely, from the values thus obtained for the coefficients γ_λ through the equations (43) and (44) in the reverse order, we should evidently arrive at the original coefficients c in question. Also the remaining coefficients c in the polynomials $p_0(z, u)$, $p_1(z, u)$, $\cdots$, $p_{n-2}(z, u)$ are identical with those in $T_{n-1}(z, u)$—the constant term c_{00} in the last named function being of course left out of consideration—so that the polynomials $p_0(z, u)$, $\cdots$, $p_{n-2}(z, u)$ in (49), regarded as a reduced form of the expression (39), are those for which the expression (49) was originally assumed to be finite at ∞. It follows therefore that the uniquely determined expression (39) is finite for $z = \infty$.

To recapitulate—we have proved in the above that in the expression (39) the coefficients γ_λ and the coefficients in the polynomial $T_{n-1}(z, u)$ can be chosen, and that uniquely, in such way that this expression does not become infinite for $z = \infty$ in case we have so selected our $d + p$ points (a_λ, b_λ) that they do not lie on a curve of degree $n - 3$, and we have furnished a method for determining the coefficients in question.

We have assumed from the outset that the d double points are included under the $d + p$ points (a_λ, b_λ) in the expression (39). It might however be remarked in this connection that all our reasoning above in regard to this expression holds good for any $d + p$ arbitrary points. That we can choose $d + p$ points (a_λ, b_λ) which do not lie on a curve of degree $n - 3$ is self-evident, for $d + p = \frac{1}{2}(n - 1)(n - 2)$ is the number of coefficients in the general equation to a curve of degree $n - 3$.

That however, among $d + p$ points so chosen we may include the d double points is not self-evident. It may however be proved that the passage through the double points imposes d conditions on our curve, or in other words that the conditions of adjointness are independent of one another.

I shall not stop here to prove the theorem in question but merely remark that the methods of the present paper furnish an extremely simple algebraic proof of it, which I propose to give in another paper at an early date. Meantime I shall accept the principle of the independence of the conditions of adjointness as

demonstrated, and we may therefore assume that the d double points are included among the $d+p$ points (a_λ, b_λ); and in (39), whatever the values of the constants in the expression under the sign of integration, we may regard the coefficients γ_λ and the coefficients in the polynomial $T_{n-1}(z, u)$ in the subtracted function as so determined that the difference does not become infinite for $z = \infty$.

§ 11.

Instead of (39) I shall now consider the expression

$$(68)\quad \int \left\{ \sum_\epsilon \frac{\beta_\epsilon F(a_\epsilon, u)}{(z-a_\epsilon)(u-b_\epsilon)} + T_{n-2}(z, u) \right\} dz - \left\{ \sum_{\lambda=1}^{d+p} \left(\frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} + m_\lambda \Pi(a_\lambda, b_\lambda) \right) + T_{n-1}(z, u) \right\},$$

where by $\Pi(a_\lambda, b_\lambda)$ we indicate an elementary integral of the second kind possessing a single infinity of the first order at the point (a_λ, b_λ), and where the coefficients m_λ may be supposed to be so determined that the expressions

$$\frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} + m_\lambda \Pi(a_\lambda, b_\lambda)$$

do not become infinite at the points (a_λ, b_λ).

The residues of the integrals $\Pi(a_\lambda, b_\lambda)$ we may assume to have the value 1, and the coefficients m_λ will then evidently be determined by the equations

$$\gamma_\lambda F^r_{b_\lambda}(a_\lambda, b_\lambda) + m_\lambda = 0.$$

In the case of the double points the coefficients m_λ will of course have the value 0.

The expression (68) does not possess infinities other than those of (39) and it does not possess even these, for the infinities of the latter expression are all to be found among the points (a_λ, b_λ), for all of which (68) is finite. The expression (68) is then everywhere finite and must therefore be an integral of the first kind. On indicating this integral by I and on supposing (a_{p+1}, b_{p+1}), (a_{p+2}, b_{p+2}), $\cdots$, (a_{p+d}, b_{p+d}) to denote the d double points we may write (68) in the form:

$$(69)\quad \int \left\{ \sum_\epsilon \frac{\beta_\epsilon F(a_\epsilon, u)}{(z-a_\epsilon)(u-b_\epsilon)} + T_{n-2}(z, u) \right\} dz - \left\{ \sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} + T_{n-1}(z, u) \right\} - \sum_{\lambda=1}^{p} m_\lambda \Pi(a_\lambda, b_\lambda) = \mathrm{I}.$$

Returning now to formula (35) and substituting therein for the integral

$$\int \left\{ \sum_\epsilon \frac{\beta_\epsilon F(a_\epsilon, u)}{(z-a_\epsilon)(u-b_\epsilon)} + T_{n-2}(z, u) \right\} dz$$

the expression derived from (69), we obtain:

$$\int R(z, u)dz = \overline{G}(z, u) + \sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + T_{n-1}(z, u) + \mathrm{I} + \sum_{\lambda=1}^{p} m_\lambda \mathrm{II}(a_\lambda, b_\lambda) + \mathrm{III}.$$

Indicating the aggregate rational portion of this expression by $G(z, u)$ we have:

$$\int R(z, u)dz = G(z, u) + \mathrm{I} + \sum_{\lambda=1}^{p} m_\lambda \mathrm{II}(a_\lambda, b_\lambda) + \mathrm{III}. \tag{70}$$

Our original Abelian integral $\int R(z, u)dz$ is herewith reduced to the sum of a rational function, an integral of the first kind, a linear function of p elementary integrals of the second kind and an integral of the third kind, the several steps in the process thereto employed being all of algebraic character. From a glance at the process also, it is plain that the coefficients m_λ and the function $G(z, u)$, apart from the arbitrary constant of integration, are completely determined when the system of points $(a_1, b_1), \cdots, (a_p, b_p)$ is given.

§ 12.

We have effected the reduction of the general Abelian integral indicated in (70), on the hypothesis that the p points $(a_1, b_1), \cdots, (a_p, b_p)$ do not lie on an adjoint curve of degree $n - 3$. In the case where they do lie on an adjoint curve of degree $n - 3$ it may readily be shown that such a reduction is not in general possible.

For suppose that we have a system of p points

$$(\overline{a}_1, \overline{b}_1), \cdots, (\overline{a}_p, \overline{b}_p)$$

lying on an adjoint curve of degree $n - 3$ and that the p points

$$(a_1, b_1), \cdots, (a_p, b_p)$$

do not all lie on such a curve.

Referring now to the expression in (39), which we sought to render finite at ∞ by proper choice of the coefficients γ_λ and of the polynomial $T_{n-1}(z, u)$ and regarding in particular the formulæ in (43), (44), (49), we remark that the constants c are independent of our choice of the p points $(a_1, b_1), \cdots, (a_p, b_p)$. From the form in (49), namely, we see that they depend for their values on the integral

$$\int \{z^{-1}q_1(z, u) + z^{-2}q_2(z, u) + \cdots + z^{-(n-1)}q_{n-1}(z, u) + Q(z, u)\}dz$$

alone, or what is the same thing, on the integral

$$\int \left\{ \sum_c \frac{\beta_c F(a_c, u)}{(z - a_c)(u - b_c)} + T_{n-2}(z, u) \right\} dz$$

in (39), independently of what our choice of the system of p points (a_1, b_1), $\cdots$, (a_p, b_p) may happen to be.

From the equations (44) again the quantities $\gamma_{i\kappa}$ depend upon the quantities c alone and are therefore also independent of the choice of our system of points (a_1, b_1), $\cdots$, (a_p, b_p).

In our system of equations (43), then, the quantities $\gamma_{i\kappa}$ on the left have the same values whatever the p points (a_1, b_1), $\cdots$, (a_p, b_p) may be and depend solely on the integral in (39).

To prove that we cannot employ the system of p points $(\bar{a}_1, \bar{b}_1)$, $\cdots$, $(\bar{a}_p, \bar{b}_p)$ for the reduction of the general Abelian integral it will suffice to show in some particular case that the quantities $\gamma_{i\kappa}$ corresponding to an integral of the type of that in (39) are such that the system of equations

$$\gamma_{i\kappa} = \sum_{\lambda=1}^{d-p} \gamma_\lambda \bar{a}_\lambda^i \bar{b}_\lambda^\kappa \tag{71}$$

cannot be satisfied. The d double points for convenience I shall designate indifferently by (a_{p+1}, b_{p+1}), $\cdots$, (a_{p+d}, b_{p+d}) or $(\bar{a}_{p+1}, \bar{b}_{p+1})$, $\cdots$, $(\bar{a}_{p+d}, \bar{b}_{p+d})$. Consider now in particular the function

$$\sum_{\lambda=1}^{p+d} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + \sum_{\lambda=1}^{p} m_\lambda \Pi(a_\lambda, b_\lambda),$$

where we suppose the coefficients m_λ to be so determined that the function does not become infinite for any of the points (a_λ, b_λ). The derivative of this function with regard to z will not become infinite to an order as high as 1 for any finite point, and for the points at ∞ it will evidently not become infinite to an order higher than $n - 3$ or, what is sufficient for our purpose, it will not become infinite at ∞ to an order higher than $n - 2$. This derivative therefore will plainly be of that type of function which appears under the sign of integration in (39).

On properly determining the coefficients β and the polynomial $T_{n-2}(z, u)$, then, we may write:

$$\sum_c \frac{\beta_c F(a_c, u)}{(z - a_c)(u - b_c)} + T_{n-2}(z, u) = \frac{d}{dz} \left\{ \sum_{\lambda=1}^{p+d} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + \sum_{\lambda=1}^{p} m_\lambda \Pi(a_\lambda, b_\lambda) \right\}.$$

The difference

$$(72)\qquad \int \left\{ \sum_{e} \frac{\beta_e F(a_e, u)}{(z-a_e)(u-b_e)} + T_{n-2}(z, u) \right\} dz + \sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)}$$

is then evidently an expression of the type (39) which does not become infinite for $z = \infty$.

Here the quantities $\gamma_{i\varkappa}$ corresponding to the integral in (72) are given by the system of equations:

$$\gamma_{i\varkappa} = \sum_{\lambda=1}^{d+p} \gamma_\lambda a_\lambda^i b_\lambda^\varkappa$$

and on starting out from a properly chosen set of values for the quantities γ_λ we could arrive at any arbitrarily assigned system of values for the quantities $\gamma_{i\varkappa}$, since for the determinant of the coefficients of the quantities γ_λ to the right of these equations we have $|a_\lambda^i b_\lambda^\varkappa| \neq 0$. To this assigned system of values for the quantities $\gamma_{i\varkappa}$ will correspond a certain set of values γ_λ and therewith a certain integral of the type in (72).

Now for the reduction of this integral the system of p points $(\bar{a}_1, \bar{b}_1), \cdots, (\bar{a}_p, \bar{b}_p)$ will not serve if values have been assigned to the quantities $\gamma_{i\varkappa}$ for which the equations (71) cannot be solved. Such values, however, it is always possible to assign to these quantities since for the determinant of the coefficients of the quantities $\bar{\gamma}$ to the right of these equations we have $|\bar{a}_\lambda^i, \bar{b}_\lambda^\varkappa| = 0$. It follows then that a system of p points $(\bar{a}_1, \bar{b}_1), \cdots, (\bar{a}_p, \bar{b}_p)$ which lie on an adjoint curve of degree $n-3$ cannot serve for the reduction of the general Abelian integral.

It will be remembered that in obtaining the reduced form on the right of (70), we assumed that no one of the p points $(a_1, b_1), \cdots, (a_p, b_p)$ was a double point. The only limitation otherwise placed upon our choice of these points was that they should not all be zeros of one and the same adjoint curve of degree $(n-3)$.

It may however be remarked that in obtaining the reduced form in question, we might also have included under the p elementary integrals of the second kind integrals corresponding to the double points, or more accurately speaking, to certain of the branches through the double points.

To have done this it would have been necessary to introduce some modifications into our reasoning which would have added to the length of the same. The modifications in question however offer no special difficulty and we may in all cases make the general statement that any Abelian integal can be reduced to the sum of a rational function of (z, u), of an integral of the first kind, of an expression linear in p elementary integrals of the second kind, and of an integral of the third kind, where in our choice of the p elementary integrals of the second kind we are subject to this one limitation only, that an adjoint curve of

degree $n - 3$ shall not exist which—over and above the zeros already implied in the fact of adjointness—possesses zeros corresponding to the several points (a_λ, b_λ) (and more precisely in the case of double points, corresponding to the individual branches), for which our p elementary integrals have been respectively constructed.

To cover all cases we should have to modify the notation employed in the formula (70) to designate an elementary integral of the second kind, as the notation $\Pi(a_\lambda, b_\lambda)$ in the case of a double point does not distinguish between the elementary integrals corresponding to the separate branches, either or both of which may happen to be included in our system of p elementary integrals of the second kind.

CHICAGO, *November* 27, 1900.

UEBER FLÄCHEN VON CONSTANTER GAUSSSCHER KRÜMMUNG*

VON

DAVID HILBERT

Ueber Flächen von negativer constanter Krümmung.

Nach BELTRAMI† verwirklicht eine Fläche von negativer constanter Krümmung ein Stück einer LOBATSCHEFSKIJschen (nicht-EUKLIDischen) Ebene, wenn man als Gerade der LOBATSCHEFSKIJschen Ebene die geodätischen Linien der Fläche von constanter Krümmung betrachtet und als Längen und Winkel in der LOBATSCHEFSKIJschen Ebene die wirklichen Längen und Winkel auf der Fläche nimmt. Unter den bisher untersuchten Flächen negativer constanter Krümmung finden wir *keine*, die sich stetig und mit stetiger Aenderung ihrer Tangentialebene in der Umgebung jeder Stelle überall hin ausdehnt; vielmehr besitzen die bekannten Flächen negativer constanter Krümmung singuläre Linien, über die hinaus eine stetige Fortsetzung mit stetiger Aenderung der Tangentialebene nicht möglich ist. Aus diesem Grunde gelingt es mittelst keiner der bisher bekannten Flächen negativer constanter Krümmung, die *ganze* LOBATSCHEFSKIJsche Ebene zu verwirklichen, und es erscheint uns die Frage von principiellem Interesse, *ob die* GANZE LOBATSCHEFSKIJ*sche Ebene überhaupt nicht durch eine analytische Fläche negativer constanter Krümmung auf die* BELTRAMI*sche Weise zur Darstellung gebracht werden kann.*

Um diese Frage zu beantworten, gehen wir von der Annahme einer analytischen Fläche der negativen constanten Krümmung -1 aus, die im Endlichen überall sich regulär verhält und keine singulären Stellen aufweist; wir werden dann zeigen, dass diese Annahme auf einen Widerspruch führt. Eine solche Fläche, wie wir sie annehmen wollen, ist durch folgende Aussagen vollständig charakterisirt:

> Jede im Endlichen gelegene Verdichtungstelle von Punkten der Fläche ist ebenfalls ein Punkt der Fläche.

Bedeutet O irgend einen Punkt der Fläche, so ist es stets möglich, das rechtwinklige Coordinatenkreuz x, y, z so zu legen, dass O der Anfangspunkt des Coordinatensystems wird und die Gleichung der Fläche in der Umgebung dieses Punktes O wie folgt lautet:

* Presented to the Society, October 27, 1900. Received for publication October 9, 1900.

† Giornale di Matematiche, Bd. 6, 1868.

$$z = ax^2 + by^2 + \mathfrak{P}(x, y),$$

wo die Constanten a, b die Relation

$$4ab = -1$$

befriedigen und die Potenzreihe $\mathfrak{P}(x, y)$ nur Glieder dritter oder höherer Dimension in x, y enthält. Offenbar ist dann die z-Axe die Normale der Fläche und die x- und y-Axe geben die Richtungen an, die durch die Hauptkrümmungen der Fläche bestimmt sind.

Die Gleichung

$$ax^2 + by^2 = 0$$

bestimmt die beiden Haupttangenten der Fläche durch den Punkt O in der xy-Ebene; dieselben sind daher stets von einander getrennt und geben die Richtungen an, in denen die beiden Asymptotencurven der Fläche durch den beliebigen Punkt O verlaufen. Jede dieser Asymptotencurven gehört einer einfachen Schaar von Asymptotencurven an, die die ganze Umgebung des Punktes O auf der Fläche regulär und lückenlos überdecken. Verstehen wir daher unter u, v genügend kleine Werthe, so können wir gewiss folgende Construktion ausführen. Wir tragen auf einer der beiden durch O gehenden Asymptotencurven den Parameterwerth u von O als Länge ab, ziehen durch den erhaltenen Endpunkt die andere mögliche Asymptotencurve und tragen auf dieser den Parameterwerth v ab: der nun erhaltene Endpunkt ist ein Punkt der Fläche, der durch die Parameterwerthe u, v eindeutig bestimmt ist. Fassen wir demgemäss die rechtwinkligen Coordinaten x, y, z der Fläche als Functionen von u, v auf, indem wir setzen:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \tag{1}$$

so sind diese jedenfalls für genügend kleine Werthe von u, v reguläre analytische Functionen von u, v.

Die bekannte Theorie der Flächen von der constanten Krümmung -1 liefert uns ferner die folgenden Thatsachen:

Bedeutet φ den Winkel zwischen den beiden Asymptotencurven durch den Punkt u, v, so erhalten die drei Fundamentalgrössen der Fläche die Werthe:

$$e \equiv \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = 1,$$

$$f \equiv \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial x}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = \cos\varphi,$$

$$g \equiv \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 1;$$

und mithin wird das Quadrat der Ableitung der Bogenlänge einer beliebigen

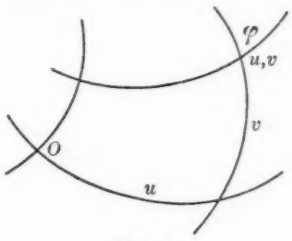

FIG. 1.

Curve auf der Fläche nach einem Parameter t von der Form:

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{du}{dt}\right)^2 + 2\cos\varphi\,\frac{du}{dt}\frac{dv}{dt} + \left(\frac{dv}{dt}\right)^2. \tag{2}$$

Der Winkel φ genügt als Function von u, v der partiellen Differentialgleichung

$$\frac{\partial^2\varphi}{\partial u\partial v} = \sin\varphi\,. \tag{3}$$

Die Formel (2) beweist den bekannten Satz:*

> In jedem Vierecke, das von vier Asymptotencurven unserer Fläche gebildet wird, sind die gegenüberliegenden Bogen einander gleich.

Die Formel (3) gestattet die Berechnung des Flächeninhaltes eines von Asymptotencurven gebildeten Viereckes mittelst seiner Winkel; DARBOUX† ist auf diesem Wege zu dem folgenden Satze gelangt:

> Der Flächeninhalt eines aus Asymptotencurven gebildeten Viereckes auf unserer Fläche ist gleich der Summe der Winkel des Viereckes vermindert um 2π.

Die Formeln (1) liefern eine Parameterdarstellung unserer Fläche, bei welcher die Coordinatenlinien

$$u = \text{const.}, \quad v = \text{const.}$$

die Asymptotencurven sind. Nach den obigen Ausführungen erweisen sich die rechtwinkligen Coordinaten x, y, z gewiss für genügend kleine Werthe von u, v als umkehrbar eindeutige Functionen der Variabeln u, v d. h. die Formeln (1) vermitteln jedenfalls die umkehrbar eindeutige Abbildung eines *Stückes der* uv-Ebene in der Umgebung des Punktes $u = 0$, $v = 0$ auf ein *Stück* unserer Fläche in der Umgebung des Punktes O. Unsere Aufgabe besteht darin, die

* DINI, Annali di Mathematica, Bd. 4, 1870, S. 175.
DARBOUX, Leçons sur la théorie générale des surfaces, Bd. 3, No. 773.
BIANCHI, Lezioni di geometria differenziale, § 67.

† loc. cit., Bd. 3, No. 773.

gesammte Abbildung der uv-Ebene auf unsere Fläche zu untersuchen, welche durch die analytische Fortsetzung der Formeln (1) erhalten wird.

Fassen wir irgend eine Asymptotencurve unserer Fläche ins Auge, so erkennen wir sofort, dass dieselbe im Endlichen keinen singulären Punkt haben und daher auch nirgends aufhören darf; denn bei Annahme einer solchen singulären Stelle könnten wir in dieselbe den Punkt O verlegen und dies gäbe einen Widerspruch mit unseren früheren Ausführungen, wonach durch O stets zwei reguläre Asymptotencurven hindurchlaufen und eine genügend kleine Umgebung des Punktes O auf unserer Fläche durch regulär verlaufende Asymptotencurven lückenlos erfüllt wird.

Aus diesem Umstande entnehmen wir die analytische Thatsache, dass die Functionen x, y, z für alle reellen u, v eindeutig und unbegrenzt fortsetzbar sind. Um dies sicher zu erkennen, tragen wir vom Punkte O aus auf der Asymptotencurve $v = 0$ die Länge u nach der einen oder anderen Richtung hin, je nachdem u positiv oder negativ ist, ab, ziehen durch den erhaltenen Endpunkt die andere Asymptotencurve, tragen dann auf dieser die Länge v nach der einen oder anderen Richtung hin, je nachdem v positiv oder negativ ausfällt, ab und ertheilen endlich dem so erhaltenen Endpunkte, der die rechtwinkligen Coordinaten x, y, z haben möge, die Parameterwerthe u, v. Auf diese Weise ist jedem Punkte der uv-Ebene jedenfalls ein bestimmter Punkt unserer Fläche zugeordnet und die Funktionen x, y, z die diese Zuordnung vermitteln sind eindeutige für alle reellen Variabeln u, v definirte und reguläre analytische Functionen.

Auch zeigt sich sofort, dass umgekehrt jedem Punkte unserer Fläche mindestens ein Werthepaar u, v entspricht. Um dies einzusehen, bezeichnen wir diejenigen Punkte, deren Coordinaten durch Functionswerthe

$$x(u, v), \quad y(u, v), \quad z(u, v)$$

dargestellt werden, mit P, dagegen die Punkte der Fläche, die durch unsere Abbildung (1) nicht betroffen werden, mit Q. Würden nun im Endlichen ein oder mehrere Punkte Q vorhanden sein, so müsste es gewiss mindestens einen Punkt A auf der Fläche geben, in dessen beliebiger Nähe sowohl Punkte P als auch Punkte Q gelegen sind.

Nach den früheren Ausführungen existiren nun für die Umgebung des Punktes A zwei Schaaren von Asymptotencurven, deren jede diese Umgebung einfach und lückenlos überdeckt. Unter diesen Asymptotencurven muss nothwendig mindestens eine solche vorhanden sein, die sowohl einen Punkt P als auch einen Punkt Q enthält. In der That, fassen wir eine der beiden durch A hin durchgehenden Asymptotencurven ins Auge und nehmen wir an dieselbe bestände aus lauter Punkten P (bez. Q), so würden die Asymptotencurven derjenigen Schaar, zu welcher jene erstere Asymptotencurve nicht gehört, mindestens je einen Punkt P (bez. Q) nämlich den Schnittpunkt mit der ersteren Asymptotencurve enthal-

ten. Die sämmtlichen Curven dieser Schaar können aber gewiss nicht aus lauter Punkten P (bez. Q) bestehen, da sonst die ganze Umgebung von A nur Punkte P (bez. Q) enthielte.

Es sei nun l die Länge eines Stückes einer Asymptotencurve, deren Anfangspunkt ein Punkt P und deren Endpunkt ein Punkt Q sein möge. Fassen wir die beiden durch den Anfangspunkt P laufenden Asymptotencurven der Fläche ins Auge, so ist jenes Stück von der Länge l nothwendig die Fortsetzung einer dieser beiden Asymptotencurven, und wenn daher u, v die Coordinaten des Anfangspunktes P sind, so wird der Endpunkt jenes Curvenstückes entweder durch die Parameterwerthe $u \pm l$, v oder u, $v \pm l$ dargestellt — entgegen unserer Annahme derzufolge der Endpunkt Q nicht durch die Formeln (1) darstellbar sein sollte.

Damit ist bewiesen worden, dass durch die Formeln (1) die ganze Fläche zur Darstellung gebracht wird, wenn u, v alle reellen Zahlenwerthe durchläuft.

Endlich ist es für unsere Untersuchung nothwendig einzusehen, dass die Formeln (1) jeden Punkt der Fläche nur durch *ein* Werthepaar u, v darstellen, d. h. dass die gefundene Abbildung (1) unserer Fläche auf die uv-Ebene nicht blos für genügend kleine Gebiete sondern im Ganzen genommen eine *umkehrbar*-eindeutige sein muss.

Wir beweisen zu dem Zwecke der Reihe nach folgende Sätze:

1. *Es giebt auf unserer Fläche keine geschlossene, d. h. in sich zurückkehrende Asymptotencurve.*

Zum Beweise nehmen wir im Gegentheil an, es sei eine solche Asymptotencurve auf unserer Fläche vorhanden. Wir construire durch jeden Punkt

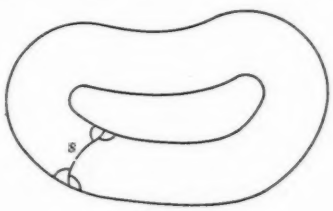

FIG. 2.

derselben die andere Asymptotencurve und tragen auf diesen Curven stets ein Stück s nach derselben Seite hin. Die erhaltenen Endpunkte werden dann entweder eine in sich zurücklaufende Asymptotencurve bilden, oder die Endpunkte des Stückes s beschreiben erst nach zweimaligem Durchlaufen der Grundcurve eine in sich zurückkehrende Asymptotencurve—ein Fall, der eintreten könnte, wenn unsere Fläche eine sogenannte Doppelfläche wäre. Fassen wir nun eine derjenigen Asymptotencurven von der Länge s ins Auge, die uns vorhin zur Construction der neuen geschlossenen Asymptotencurve diente, so bildet dieselbe

doppelt gerechnet zusammen mit den beiden geschlossenen Asymptotencurven ein Asymptotenviereck, dessen Winkelsumme offenbar genau gleich 2π ist. Diese Thatsache aber steht im Widerspruch zu dem vorhin angeführten Satze, wonach der Inhalt eines Asymptotencurvenvierecks stets gleich dem Ueberschuss der Summe seiner Winkel über 2π ist und dieser Ueberschuss daher nothwendig positiv sein muss.

2. *Irgend zwei durch einen Punkt gehende Asymptotencurven schneiden sich in keinem anderen Punkt unserer Fläche.*

Wir denken uns eine Asymptotencurve a nach beiden Richtungen hin ins Unendliche verlängert und dann durch einen Punkt P_0 derselben nach einer Seite die andere Asymptotencurve b gezogen. Nehmen wir dann im Gegensatz zu unserer Behauptung an, dass diese Asymptotencurve b die ursprüngliche a zum ersten Mal im Punkt P_1 schnitte, so sind die folgenden zwei Fälle denkbar:

Erstens: die Asymptotencurve b könnte so verlaufen, dass sie in P_1 von derselben Seite der Asymptotencurve a her eintritt, als sie dieselbe verlassen hat;

Zweitens: die Asymptotencurve b könnte derart verlaufen, dass sie von der anderen Seite der ursprünglichen Asymptotencurve a herkommt und mithin nach Verlassen des Schnittpunktes P_1 nach der nämlichen Seite der Asymptotencurve a gerichtet ist, wie anfänglich, als sie vom Punkte P_0 ausging.

Wir wollen zeigen, dass beide Fälle unmöglich sind. Was den *ersten* Fall betrifft, so bezeichnen wir die Länge der Strecke P_0P_1 auf a mit l und die Mitte dieser Strecke mit Q. Sodann denken wir uns durch jeden Punkt der

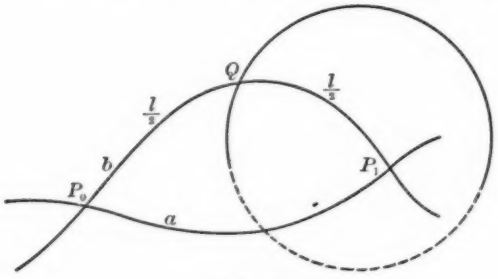

FIG. 3.

Asymptotencurve a die andere Asymptotencurve gezogen und nach der Seite, nach welcher hin die fragliche Strecke P_0P_1 auf b liegt, die Länge $\frac{1}{2}l$ abgetragen. Aus den Punkten P_0 und P_1 der Curve a erhalten wir auf diese Weise den nämlichen Punkt Q als Endpunkt. Die sämmtlichen erhaltenen Endpunkte bilden mithin eine Asymptotencurve, welche durch den Punkt Q geht und zu demselben Punkte Q in der nämlichen Richtung zurückkehrt. Dies ist unmöglich, da es nach 1. auf unserer Fläche keine geschlossene Asymptotencurve giebt.

Damit ist gezeigt, dass der erste Fall nicht stattfinden kann. Aber auch der *zweite* Fall ist unmöglich. Verliefe nämlich die Asymptotencurve in der Weise, dass sie nach Ueberschreitung des Schnittpunktes P_1 die nämliche Richtung aufweist, wie früher in P_0, so könnten wir die Fortsetzung dieses Stückes P_0P_1 der Asymptotencurve b über P_1 hinaus offenbar dadurch erhalten, dass wir von P_0 ausgehend durch jeden Punkt des Stückes P_0P_1 von b die andere Asymptotencurve construiren und auf allen diesen Asymptotencurven nach der betreffenden Seite hin das gleiche Stück P_0P_1 der Asymptotencurve a abtragen. Die erhaltenen Endpunkte bilden die Fortsetzung der Asymptotencurve b von P_1 bis zu einem Punkte P_2 auf a. Aus diesem Stücke P_1P_2 der Asymptotencurve b können wir in gleicher Weise ein neues Stück der Asymptotencurve b construiren, welches über P_2 hinaus geht und bis zu einem Punkte P_3 auf a reicht u. s. f. Auch ist klar, wie wir die Asymptotencurve b nach der anderen Richtung hin über P_0 hinaus durch die entsprechende Construction fortsetzen können und so der Reihe nach zu den Curvenstücken P_0P_{-1}, $P_{-1}P_{-2}, \cdots$ gelangen. Die Asymptotencurve b schneidet also die Asymptotencurve a in den unendlich vielen gleich weit von einander entfernten Punkten:

$$\cdots P_{-3},\ P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2,\ P_3 \cdots.$$

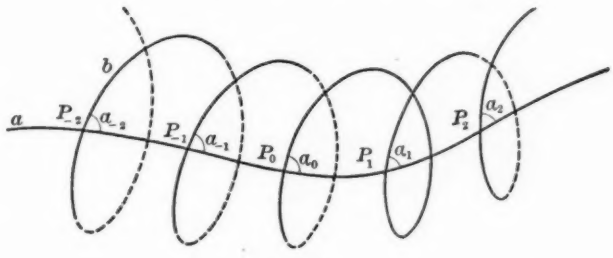

FIG. 4.

Die Winkel, die die Asymptotencurve b mit a in bestimmten Sinne in jenen Schnittpunkten bildet, bezeichnen wir bez. mit

$$\cdots \alpha_{-3},\ \alpha_{-2},\ \alpha_{-1},\ \alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3 \cdots.$$

Wir betrachten nun das Asymptotencurvenviereck $P_0P_1P_2P_1P_0$, welches von den zwei Stücken P_0P_1, P_2P_1 auf a und den zwei Curvenstücken P_1P_2, P_1P_0 auf b gebildet wird. Die vier Winkel dieses Vierecks sind

$$\alpha_0,\ \pi - \alpha_1,\ \alpha_2,\ \pi - \alpha_1,$$

und da nach dem angeführten Satze über das Asymptotencurvenviereck der Inhalt desselben gleich dem Ueberschuss der Summe seiner Winkel über 2π ist und dieser Ueberschuss daher positiv sein muss, so folgt

$$a_0 + \pi - a_1 + a_2 + \pi - a_1 > 2\pi,$$

d. h.

$$a_0 - a_1 > a_1 - a_2. \tag{4}$$

Ebenso folgt allgemein

$$a_k - a_{k+1} > a_{k+1} - a_{k+2}, \qquad (k = 0, \pm 1, \pm 2, \cdots). \tag{5}$$

Wegen der obigen Ungleichung (4) können jedenfalls $a_0 - a_1$ und $a_1 - a_2$ nicht zugleich 0 sein; wir dürfen die Annahme

$$a_0 - a_1 \neq 0$$

treffen. Aus (5) folgen die Ungleichungen:

$$a_{-p} - a_{-p+1} > a_0 - a_1 \qquad (p = 1, 2, 3, \cdots), \tag{6}$$

$$a_0 - a_1 \quad > a_p - a_{p+1}$$

oder

$$a_{p+1} - a_p \quad > a_1 - a_0 \qquad (p = 1, 2, 3, \cdots). \tag{7}$$

Bilden wir die Ungleichungen (6) und (7) für $p = 1, 2, 3, \cdots, n$, so folgt durch Addition derselben leicht

$$a_{-n} > a_0 + n(a_0 - a_1),$$

$$a_{n+1} > a_1 + n(a_1 - a_0).$$

Fällt nun $a_0 - a_1 > 0$ aus, so ist für genügend grosse Werthe von n jedenfalls die erstere dieser beiden Gleichungen unmöglich, da die Winkel a_k sämmtlich kleiner als π sind; fällt dagegen $a_1 - a_0 > 0$ aus, so folgt aus demselben Grunde für genügend grosse Werthe von n die Unmöglichkeit der letzteren Gleichung.

Die Asymptotencurve b darf daher auf keine der beiden angenommenen Arten verlaufen und mithin ist der Beweis für 2. vollständig erbracht.

3. *Eine Asymptotencurve unserer Fläche durchsetzt sich selbst an keiner Stelle, d. h. sie besitzt keinen Doppelpunkt.*

Zum Beweise nehmen wir im Gegentheil an, es existire eine Asymptotencurve mit einem Doppelpunkt; dann verlegen wir den Anfangspunkt der krummlinigen Coordinaten u, v in diesen Doppelpunkt und wählen die beiden Zweige der Curvenschleife zu Coordinatenlinien, nach der Schleife hin den positiven Sinn gerechnet.

Wir ziehen jetzt vom Doppelpunkte (0, 0) oder vom Punkte $(-s, 0)$ beginnend auf der u-Coordinate durch jeden Punkt der Schleife die andere Asymptotencurve und tragen auf dieser nach der positiven Seite hin eine Strecke s ab: wählen wir diese Strecke s genügend klein, so werden die sämmtlichen erhaltenen Endpunkte nach einem oben angeführten Satze über das Asymptotencurvenviereck wiederum eine Asymptotencurve bilden. Diese Asymptotencurve geht vom

Punkte $(0, s)$ bez. $(-s, s)$ aus, durchsetzt sich selbst im Punkte (s, s) bez. $(-s, s)$ und endigt im Punkte $(s, 0)$ bez. $(-s, 0)$. Wir sehen also, dass die eben construirte Asymptotencurve die ursprüngliche Asymptotencurve in zwei

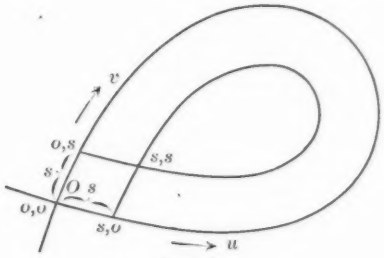

FIG. 5.

verschiedenen Punkten $(0, s)$ und $(s, 0)$ bez. $(0, s)$ und $(-s, 0)$ schneidet; dies ist nach 2. unmöglich.

4. *Wenn wir durch jeden Punkt einer Asymptotencurve a die andere Asymptotencurve ziehen und auf dieser nach der nämlichen Seite hin eine bestimmte Strecke s abtragen, so bilden die erhaltenen Endpunkte eine neue Asymptotencurve b, die die ursprüngliche Asymptotencurve a an keiner Stelle schneidet.*

Denn wäre P ein Schnittpunkt der Asymptotencurve b mit der ursprünglichen a, und tragen wir von P aus auf der Asymptotencurve b die Strecke s nach der betreffenden Seite von a hin ab, so müsste der weitere durch den entstehenden Endpunkt Q hindurchgehende Asymptotencurvenzweig ebenfalls zur Asympto-

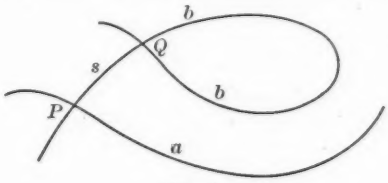

FIG. 6.

tencurve b gehören und mithin wäre Q ein Doppelpunkt der Asymptotencurve b; das Auftreten eines Doppelpunktes ist aber nach 3. unmöglich.

Aus den Sätzen 1. bis 4. können wir sofort diese Schlussfolgerungen ziehen:

Die sämmtlichen Asymptotencurven unserer Fläche zerfallen in zwei Schaaren. Irgend zwei derselben Schaar angehörende Asymptotencurven schneiden sich nicht; dagegen schneiden sich je zwei Asymptotencurven, die verschiedenen Schaaren angehören, stets in einem und nur einem Punkte der Fläche.

Die Coordinatenlinien $u = 0$, $v = 0$ sind zwei Asymptotencurven die verschiedenen Schaaren angehören. Wegen der Bedeutung der Coordinaten u, v als Längen gewisser Coordinatenabschnitte entnehmen wir aus den eben ausge-

sprochenen Thatsachen zugleich, dass zu bestimmt gegebenen Werthen von u, v stets nur ein Punkt unserer Fläche gehört d. h. *die zu untersuchende Abbildung* (1) *unserer Fläche auf die uv-Ebene ist nothwendig eine umkehrbar eindeutige. Insbesondere folgt hieraus, dass unsere Fläche einen einfachen Zusammenhang besitzt und keine Doppelfläche ist.*

Nachdem wir zu dieser wichtigen Einsicht gelangt sind, berechnen wir den gesammten Inhalt unserer Fläche auf zwei Wegen; wir werden dadurch zu einem Widerspruch gelangen.

Der erstere Weg ist der folgende. Wir betrachten auf unserer Fläche dasjenige aus Asymptotencurven gebildete Viereck, dessen Ecken durch die Coordinaten

$$u,\ v; \qquad -u,\ v; \qquad -u,\ -v; \qquad u,\ -v$$

bestimmt sind. Da jeder Winkel dieses Vierecks $< \pi$ sein muss, so ist die Summe der Winkel des Viereckes jedenfalls $< 4\pi$ und der Inhalt des Viereckes d. h. der Ueberschuss der Summe seiner Windel über 2π ist mithin nothwendig $< 2\pi$. Lassen wir nun die Werthe von u, v unbegrenzt wachsen, so kommt jeder bestimmte Punkt der Fläche sicher einmal im Inneren eines Viereckes zu liegen und bleibt dann im Inneren aller weiteren Vierecke, so dass das unbegrenzt wachsende Viereck schliesslich die ganze Oberfläche umfasst. Wir entnehmen daraus, dass der Gesammtinhalt unserer Fläche $\leqq 2\pi$ sein muss.

Andrerseits betrachten wir die geodätischen Linien auf unserer Fläche. Wegen der negativen Krümmung unserer Fläche ist jede geodätische Linie zwischen irgend zweien ihrer Punkte gewiss kürzeste Linie, d. h. von kleinerer Länge als jede andere Linie, die auf der Fläche zwischen den nämlichen zwei Punkten verläuft und sich durch stetige Aenderung in die geodätische Linie überführen lässt. Wir fassen nun irgend zwei vom Punkt O ausgehende geodätische Linien auf unserer Fläche ins Auge und nehmen an, dieselben schnitten sich noch in einem anderen Punkt P der Fläche. Da nach dem oben Bewiesenen unsere Fläche einen einfachen Zusammenhang besitzt, so lässt sich jede dieser beiden geodätischen Linien $O\,P$ in die andere durch stetige Veränderung überführen; es müsste also nach dem eben Ausgeführten jede derselben kürzer sein als die andere, was nicht möglich ist. Unsere Annahme der Existenz eines Schnittpunktes P ist also zu verwerfen. Durch die nämlichen Schlüsse erkennen wir auch, dass eine geodätische Linie unserer Fläche weder sich durchsetzen noch in sich selbst zurücklaufen darf.

Denken wir uns nun auf allen von O ausgehenden geodätischen Linien die gleiche Länge r abgetragen, so bilden die erhaltenen Endpunkte eine geschlossene doppelpunktslose Curve auf unserer Fläche. Das von dieser Curve umspannte Gebiet besitzt nach den bekannten Formeln der LOBATSCHEFSKIJschen Geometrie den Flächeninhalt

$$\pi\left(e^{\frac{r}{2}} - e^{-\frac{r}{2}}\right)^2.$$

Da dieser Ausdruck für unendlich wachsende Werthe von r selbst über alle Grenzen wächst, so entnehmen wir hieraus, dass auch der Gesammtinhalt unserer Fläche unendlich gross sein müsste. Diese Folgerung steht im Widerspruch mit der vorhin bewiesenen Thatsache, wonach jener Inhalt stets $\leqq 2\pi$ ausfallen sollte. Wir sind daher gezwungen, unsere Grundannahme zu verwerfen, d. h. wir erkennen, *dass es eine singularitätenfreie und überall regulär analytische Fläche von constanter negativer Krümmung nicht giebt. Insbesondere ist* daher auch *die zu Anfang aufgeworfene Frage zu verneinen, ob auf die* BELTRAMIsche *Weise die* GANZE LOBATSCHEFSKIJsche *Ebene durch eine regulär analytische Fläche im Raume sich verwirklichen lässt.*

Ueber Flächen von positiver constanter Krümmung.

Wir gingen zu Anfang dieser Untersuchung aus von der Frage nach einer Fläche der negativen constanten Krümmung, die überall im Endlichen regulär analytisch verläuft, und gelangten zu dem Resultate, dass es eine solche Fläche nicht giebt. Wir wollen nunmehr mittelst der entsprechenden Methode die gleiche Frage für die positive constante Krümmung behandeln. Offenbar ist die Kugel eine geschlossene singularitätenfreie Fläche der positiven constanten Krümmung, und nach dem von H. LIEBMANN* geführten Beweise giebt es auch keine andere geschlossene Fläche von derselben Eigenschaft. Diese Thatsache nun wollen wir aus einem Satze herleiten der von einem beliebigen singularitätenfreien Stücke einer Fläche der positiven constanten Krümmung gilt und folgendermassen lautet.

Auf einer analytischen Fläche der positiven constanten Krümmung $+1$ *sei ein singularitätenfreies einfach oder mehrfach zusammenhängendes Gebiet im Endlichen abgegrenzt: denken wir uns dann in jedem Punkte dieses Gebietes sowie in den Randpunkten desselben die beiden Hauptkrümmungsradien der Fläche construirt, so wird das Maximum der grösseren und folglich auch das Minimum der kleineren der beiden Hauptkrümmungsradien gewiss in keinem Punkte angenommen, der im Inneren des Gebietes liegt — es sei denn unsere Fläche ein Stück der Kugel mit dem Radius* 1.

Zum Beweise bedenken wir zunächst, dass wegen unserer Voraussetzung das Product der beiden Hauptkrummungsradien überall $= 1$ und daher der grössere der beiden Hauptkrümmungsradien stets $\geqq 1$ sein muss. Aus diesem Grunde ist das Maximum der grösseren Hauptkrümmungsradien offenbar nur dann $= 1$, wenn beide Hauptkrümmungsradien in jedem Punkte unseres Flächenstückes $= 1$ sind. In diesem besonderen Falle ist jeder Punkt des Flächenstückes ein Nabelpunkt, und man schliesst dann leicht in bekannter Weise, dass das Flächenstück ein Stück der Kugel mit dem Radius 1 sein muss.

*Göttinger Nachrichten, 1899, S. 44.

Nunmehr sei das Maximum der grösseren der beiden Hauptkrümmungsradien unserer Fläche > 1; dann nehmen wir im Gegensatz zu der Behauptung an, es gäbe *im Inneren* des Flächenstückes einen Punkt O, in welchem jenes Maximum stattfinde. Da dieser Punkt O gewiss kein Nabelpunkt sein kann und überdies ein regulärer Punkt unserer Fläche ist, so wird die Umgebung dieses Punktes lückenlos und einfach von jeder der beiden Schaaren von Krümmungslinien der beiden Flächen bedeckt. Benutzen wir diese Krümmungslinien als Coordinatenlinien und den Punkt O selbst als Anfangspunkt des krummlinigen Coordinatensystems, so gelten nach der bekannten Theorie der Flächen positiver constanter Krümmung die folgenden Thatsachen.*

Es bedeute r_1 den grösseren der beiden Hauptkrümmungsradien für den Punkt (u, v) in der Umgebung des Anfangspunktes $O = (0, 0)$; es ist in dieser Umgebung $r_1 > 1$. Man setze

$$\rho = \tfrac{1}{2} \log \frac{r_1 + 1}{r_1 - 1};$$

dann genügt die positive reelle Grösse ρ als Function von u, v der partiellen Differentialgleichung

$$\frac{\partial^2 \rho}{\partial u^2} + \frac{\partial^2 \rho}{\partial v^2} = \frac{e^{-2\rho} - e^{2\rho}}{4}. \tag{8}$$

Da bei abnehmendem r_1 die Function ρ nothwendig wächst, so muss ρ als Function von u, v an der Stelle $u = 0, v = 0$ einen Minimalwerth aufweisen, und demnach hat die Entwickelung von ρ nach Potenzen der Variabeln u, v nothwendig die Gestalt

$$\rho = a + \alpha u^2 + 2\beta uv + \gamma v^2 + \cdots,$$

wo a, α, β, γ Constante bedeuten und dabei die quadratische Form

$$\alpha u^2 + 2\beta uv + \gamma v^2,$$

für reelle u, v niemals negative Werthe annehmen darf. Aus letzterem Umstande folgen für die Constanten α und γ nothwendig die Ungleichungen:

$$\alpha \geqq 0 \text{ und } \gamma \geqq 0. \tag{9}$$

Andrerseits wollen wir die Entwicklung für ρ in die Differentialgleichung (8) einsetzen: für $u = 0, v = 0$ erhalten wir dann

$$2(\alpha + \gamma) = \frac{e^{-2a} - e^{2a}}{4}.$$

Da die Constante a den Werth von ρ im Punkte $O = (0, 0)$ darstellt und mithin

*Darboux, *Leçons sur la theorie générale des surfaces*, Bd. 3, No. 776.
Bianchi, *Lezioni di geometria differenziale*, § 264.

positiv ausfällt, so ist hier der Ausdruck rechter Hand jedenfalls < 0; die letztere Gleichung führt desshalb zu der Ungleichung

$$\alpha + \gamma < 0,$$

welche mit den Ungleichungen (9) in Widerspruch steht. Damit ist unsere ursprüngliche Annahme, wonach die Stelle des Maximums im Inneren des Flächenstückes liege als unzutreffend und mithin der oben aufgestellte Satz als richtig erkannt.

Der eben bewiesene Satz lehrt offenbar folgende Thatsache. Wenn wir aus der Kugeloberfläche ein beliebiges Stück ausgeschnitten denken und dann dieses Stück beliebig verbiegen, so findet sich das Maximum aller grösseren vorkommenden Hauptkrümmungsradien stets auf dem Rande des Flächenstückes. Eine geschlossene Fläche besitzt keinen Rand und daraus folgt, wie bereits oben bemerkt, sofort der Satz, *das eine geschlossene analytische singularitätenfreie Fläche mit der positiven constanten Krümmung* 1 *stets die Kugel mit dem Radius* 1 *sein muss.* Dieses Resultat drückt zugleich aus, dass man die Kugel als Ganzes nicht verbiegen kann, ohne den regulär analytischen Charakter der Fläche irgendwo zu stören.

GÖTTINGEN, 1900.

NOTE ON THE FUNCTIONS OF THE FORM

$$f(x) \equiv \phi(x) + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$$

WHICH IN A GIVEN INTERVAL DIFFER THE LEAST POSSIBLE FROM ZERO*

BY

H. F. BLICHFELDT

TSCHEBYCHEFF has considered the following problem: $f(x)$ is a given function of x and of n arbitrary constants $a_1, a_2, \cdots, a_n$, and is, together with its partial derivatives with respect to $x, a_1, a_2, \cdots, a_n$, continuous in the interval $a \leqq x \leqq b$; to determine the constants $a_1, a_2, \cdots, a_n$ so that the greatest value of $[f(x)]^2$, in the same interval, shall differ as little as possible from zero. †

His solution is as follows:

For any given set of values of the constants $a_1, a_2, \cdots, a_n$, let $x_1, x_2, \cdots, x_\mu$ be all the different values of x for which $[f(x)]^2$, in the interval $a \leqq x \leqq b$, reaches its greatest value L^2. Then must $x_1, x_2, \cdots, x_\mu$ evidently satisfy the two equations

$$\text{(I)} \qquad [f(x)]^2 - L^2 = 0, \quad (x-a)(x-b)f'(x) = 0.$$

If it is now possible to satisfy the μ equations

$$\text{(II)} \qquad \sum_{k=1}^{n} \frac{\partial f(x_i)}{\partial a_k} N_k = f(x_i) = \pm L \qquad (i = 1, 2, \cdots, \mu),$$

by finite, determinate values of the n quantities $N_1, \cdots, N_n$, then it will be possible to give to the constants $a_1, \cdots, a_n$ such small increments, proportional to $N_1, \cdots, N_n$, that the greatest absolute value of $f(x)$, for $a \leq x \leq b$, becomes less than L, which is taken positive.

* Presented to the Society (Chicago) April 14, 1900, under a slightly different title. Received for publication November 21, 1900.

† *Sur les questions de minima qui se rattachent à la représentation approximative des fonctions*, Pétersbourg Mémoires, series 6, vol. 7. The above statement of TSCHEBYCHEFF'S results is taken from J. BERTRAND, *Calcul Différentiel*, p. 512.

Consequently, the set of constants $a_1, \cdots, a_n$, for which the greatest absolute value of $f(x)$ in the given interval is the least possible, must be such that the μ equations (II) are inconsistent for finite values of the constants $N_1, \cdots, N_n$.

This requires, in many cases, that $\mu > n$, only. In particular, if

$$f(x) \equiv x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n,$$

and the interval is $-h \leqq x \leqq +h$, the necessary and sufficient condition imposed upon $x_1, \cdots, x_n$ is that the equations (I),

$$[f(x)]^2 - L^2 = 0, \quad (x^2 - h^2) f'(x) = 0,$$

possess $n+1$ different, common solutions. These solutions are therefore $-h$, $+h$, and all the $n-1$ roots of $f'(x) = 0$, which last are obviously double roots of $[f(x)]^2 - L^2 = 0$. Hence, as $f'(x) = nx^{n-1} + \text{etc.}$, we must have identically

$$[f(x)]^2 - L^2 = x^{2n} + \text{etc.} = (x^2 - h^2)\left(\frac{f'(x)}{n}\right)^2,$$

and therefore

$$\frac{f'(x)}{\sqrt{L^2 - [f(x)]^2}} = \frac{n}{\sqrt{h^2 - x^2}},$$

from which we get by integration,

$$f(x) = \frac{h^n}{2^{n-1}} \cos n \cos^{-1}\left(\frac{x}{h}\right) = \frac{1}{2^n}\{[x + \sqrt{x^2 - h^2}]^n + [x - \sqrt{x^2 - h^2}]^n\},$$

the values of L and the constant of integration being readily determined. The result just given is in accord with that of BERTRAND obtained (loc. cit., pp. 514–518) by lengthy considerations involving the theory of continued fractions. *

In general, several sets of constants $a_1, \cdots, a_n$ can be found to satisfy TSCHEBYCHEFF'S conditions. If the function $f(x)$ involves the constants in the following manner:

$$f(x) \equiv \phi(x) + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n,$$

the labor of selecting the required set is much simplified by the following considerations.

Let us by the "maxima" of $[f(x)]^2$ in the interval $a \leqq x \leqq b$ understand those only which are equal to L^2, the greatest value of $[f(x)]^2$ in the given interval. If we classify these maxima as *positive* or *negative* according as the corresponding value of $f(x)$ is $+L$ or $-L$, and plot the curve $y = f(x)$, *we*

* In Liouville's Journal, ser. 2, vol. 19 (1874), pp. 319-347, TSCHEBYCHEFF has solved this problem with the additional condition imposed upon $f(x)$, that it either never decreases or never increases in the given interval.

*shall find at least n alternations of the two kinds of maxima in the given interval.**

The curve $y' = \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_n$ is continuous and can be made to cross the axis of X at $n-1$ given points. If the curve $y = f(x) \equiv \phi(x) + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$, whose maxima are assumed to have their least possible absolute value, had less than n alternations of the two kinds of maxima, we could construct a curve $y' = \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_n$ which would have positive ordinates whenever $y = f(x)$ possessed negative maxima, and vice versa, at the same time making the largest of its ordinates in the given interval as small as we please. The maxima of the curve

$$y = f(x) + y' \equiv \phi(x) + (a_1 + \beta_1)x^{n-1} + (a_2 + \beta_2)x^{n-2} + \cdots + (a_n + \beta_n)$$

would thus be less in absolute value than those of $y = f(x)$, contrary to hypothesis.

For instance, the nearest approximation to $\sin x$ of the form $a_1 x + a_2$ in the interval $0 \leqq x \leqq h \leqq \pi/2$ must be such that the curve $y = \sin x - a_1 x - a_2$ shall have two positive maxima including one negative, or vice versa; there being just three maxima in the given interval, namely, for $x = 0$, $\cos^{-1} a_1$, h. These maxima, being given by

$$-a_2, \quad +\sqrt{1-a_1^2} - a_1 \cos^{-1} a_1 - a_2, \quad \sin h - a_1 h - a_2,$$

must therefore satisfy the relations:

$$(-a_2) + (+\sqrt{1-a_1^2} - a_1 \cos^{-1} a_1 - a_2) = 0,$$
$$-(-a_2) + (\sin h - a_1 h - a_2) = 0,$$

so that

$$a_1 = \frac{1}{h}\sin h, \quad a_2 = +\tfrac{1}{2}\sqrt{1-\left(\frac{\sin h}{h}\right)^2} - \frac{\sin h}{2h}\cos^{-1}\left(\frac{\sin h}{h}\right).$$

The approximation is, therefore,

$$\frac{\sin h}{h}x + \tfrac{1}{2}\sqrt{1-\left(\frac{\sin h}{h}\right)^2} - \frac{\sin h}{2h}\cos^{-1}\left(\frac{\sin h}{h}\right).$$

STANFORD UNIVERSITY, CAL.

* The writer has not access to the original memoirs of TSCHEBYCHEFF, in which this property may have been indicated.